# Event structure semantics of (controlled) reversible CCS

Eva Graversen, Iain Phillips, and Nobuko Yoshida

Imperial College London

**Abstract.** CCSK is a reversible form of CCS which is causal, meaning that actions can be reversed if and only if each action caused by them has already been reversed; there is no control on whether or when a computation reverses. We propose an event structure semantics for CCSK. For this purpose we define a category of reversible bundle event structures, and use the causal subcategory to model CCSK. We then modify CCSK to control the reversibility with a rollback primitive, which reverses a specific action and all actions caused by it. To define the event structure semantics of rollback, we change our reversible bundle event structures by making the conflict relation asymmetric rather than symmetric, and we exploit their capacity for non-causal reversibility.

## 1 Introduction

Reversible process calculi have been studied in works such as [5, 7, 10, 19]. One feature of such reversible processes is their ability to distinguish true concurrency in a way forward-only processes cannot [15]. For instance, using CCS notation, the processes $a|b$ and $a.b + b.a$ are equivalent under interleaving semantics; however in a reversible setting we can distinguish them by noting that $a|b$ allows us to perform $a$ followed by $b$ and then to reverse $a$, which is impossible for $a.b + b.a$. This motivates us to use event structures [14] to describe truly concurrent semantics of a reversible process calculus.

Two reversible forms of CCS have been proposed: RCCS [7] and CCSK [19]. RCCS creates separate memories to store past (executed) actions, while CCSK annotates past actions with keys within the processes themselves. We formulate an event structure semantics for CCSK rather than RCCS, since the semantics for past and future actions can be defined in a similar manner, rather than having to encompass both processes and memories. We note that Medić and Mezzina [12] showed that RCCS and CCSK can be encoded in each other, meaning one can use their encoding in conjunction with our event structure semantics to obtain an event structure semantics for RCCS.

Event structures have been used for modelling forward-only process calculi [2,4,21]. Cristescu et al. [6] used rigid families [3], related to event structures, to describe the semantics of R$\pi$ [5]. However, their semantics requires a process to first reverse all actions to find the original process, map this process to a rigid family, and then apply each of the reversed memories in order to reach the current state of the process. Aubert and Cristescu [1] used a similar approach to describe the semantics of RCCS processes without auto-concurrency, auto-conflict, or recursion as configuration structures. By contrast, we map a CCSK process (with auto-concurrency, auto-conflict, and recursion) with past actions directly to a (reversible) event structure in a strictly denotational fashion.

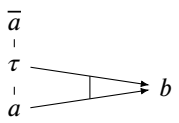
Reversible forms of prime [16], asymmetric [16], and general [18] event structures have already been defined, but the usual way of doing parallel composition of forward-only prime (PES) and asymmetric event structures (AES) [20] does not translate into a reversible setting, and general event structures are more expressive that is necessary for modelling reversible CCSK. We therefore base our semantics on a reversible form of bundle event structures (BESs) [11].

BESs were created with the specific purpose of allowing the same event to have multiple conflicting causes, thereby making it possible to model parallel composition without creating multiple copies of events. They do this by associating events with bundles of conflicting events, $X \mapsto e$, where in order for event $e$ to happen one of the events of $X$ must have already happened.

This approach can be used for modelling cases such as Example 1.1 below, where an action $a$ has multiple options for synchronisation, either of which would allow the process to continue with the action $b$. If we model each synchronisation or lack thereof as a separate event then we clearly need to let $b$ have multiple possible causes, which we can accomplish using BESs, but not using PESs.

*Example 1.1 (Process easily representable by a bundle event structure).*

The CCS process $a.b \mid \overline{a}$ can be described by a BES with the events $a, \tau, \overline{a}, b$, the bundle $\{a, \tau\} \mapsto b$, and the conflicts $a \sharp \tau$ and $\overline{a} \sharp \tau$. The process cannot be represented by a PES or AES without splitting some events into multiple events, due to $b$ having multiple possible causes.




We therefore define a category of reversible BESs (RBESs). Since the reversibility allowed in CCSK (as in RCCS) is *causal*, meaning that actions can be reversed if and only if every action caused by them has already been reversed, we use the causal subcategory of RBESs for defining a denotational semantics of CCSK.

Causal reversibility has the drawback of allowing a process to get into a loop doing and undoing the same action indefinitely; there is no control on whether or when a computation reverses. We modify CCSK to control reversibility by adding the *rollback* introduced for Roll-$\pi$ in [9]. In Roll-CCSK every action receives a tag $\gamma$, and the process only reverses when reaching a $\mathsf{roll}\ \gamma$ primitive, upon which the action tagged with $\gamma$, together with all actions caused by it, are reversed. As in Roll-$\pi$, the rollback in Roll-CCSK is *maximally permissive*, meaning that any subset of reached rollbacks may be executed, even if one of them rolls back the actions leading to another. The operational semantics of rollback work somewhat differently in Roll-CCSK from Roll-$\pi$, since Roll-$\pi$ has a set of memories describing past actions in addition to a $\pi$-calculus process, while CCSK has the past actions incorporated into the structure of the process, meaning that it is harder to know whether one has found all the actions necessary to reverse. Roll-CCSK allows recursion using binding on tags. Mezzina and Koutava [13] added rollback to a variant of CCS, though they use a set of memories to store their past actions, making their semantics closer to Roll-$\pi$.

Once a $\mathsf{roll}\ \gamma$ event has happened, we need to ensure that not only are the events caused by the $\gamma$-tagged action $a_\gamma$ able to reverse, but they cannot re-occur until the rollback is complete, at which point the $\mathsf{roll}\ \gamma$ event is reversed. This requires us to model

*asymmetric* conflict between $\mathsf{roll}\ \gamma$ and events caused by $a_\gamma$ (apart from $\mathsf{roll}\ \gamma$ itself). Asymmetric conflict is allowed in *extended* BESs (EBESs) [11]. We define a category of reversible EBESs (REBESs) and use them to give an event structure semantics of rollback. Note that we do not restrict ourselves to the causal subcategory of REBESs, since reversibility in Roll-CCSK is not necessarily causal. An action $a_\gamma$ tagged with $\gamma$ is a cause of $\mathsf{roll}\ \gamma$, but we want $a_\gamma$ to reverse before $\mathsf{roll}\ \gamma$ does.

*Contributions* We formulate reversible forms of bundle, and extended bundle event structures. We show that these form categories equipped with products and coproducts. We extend CCSK with recursion and use the category of RBESs to define its event structure semantics. We define the operational semantics of Roll-CCSK, which uses rollback to control the reversibility in CCSK, showing that our rollbacks are both sound (Theorem 6.6) and complete (Theorem 6.9) with respect to CCSK. We use the category of REBESs to define the event structure semantics of Roll-CCSK. We prove operational correspondence between the operational semantics and event structure semantics of both CCSK and Roll-CCSK (Theorems 4.10, 7.5 and 7.7).

*Outline* Section 2 recalls the semantics of CCSK. Section 3 describes RBESs and their category. Section 4 defines the event structure semantics of CCSK. Section 5 describes REBESs and their category. Section 6 introduces Roll-CCSK and its operational semantics and Section 7 uses REBESs to describe the event structure semantics of Roll-CCSK.

## 2 CCSK

CCSK was defined in [19], and distinguishes itself from most reversible process calculi by retaining the structure of the process when actions are performed, and annotating past actions with keys instead of generating memories. For instance we have $a.P \mid \overline{a}.Q \xrightarrow{\tau[n]} a[n].P \mid \overline{a}[n].Q$, with the key $n$ denoting that $a$ and $\overline{a}$ have previously communicated, and we therefore cannot reverse one without reversing the other.

We call the set of actions of CCSK $\mathcal{A}$ and let $a, b, c$ range over $\mathcal{A}$, $\alpha, \beta$ range over $\mathcal{A} \cup \overline{\mathcal{A}}$, and $\mu$ range over $\mathcal{A} \cup \overline{\mathcal{A}} \cup \{\tau\}$. We let $\mathcal{K}$ be an infinite set of communication keys and let $m, n$ range over $\mathcal{K}$.

CCSK then has the following syntax, very similar to CCS:

$$P ::= \alpha.P \mid \alpha[n].P \mid P_0 + P_1 \mid P_0 | P_1 \mid P \setminus A \mid P[f] \mid 0$$

Here $P \setminus A$ restricts communication on actions in $A \cup \overline{A}$ and $P[f]$ applies a function $f : \mathcal{A} \to \mathcal{A}$ to actions done by $P$.

Table 1 shows the forwards rules of the operational semantics of CCSK. As CCSK is causal, the reverse rules can be derived from these. We use $\rightsquigarrow$ to denote a reverse action, $\mathsf{std}(P)$ to denote that $P$ is a standard process, meaning it contains no past actions, and $\mathsf{fsh}[n](P)$ to denote that the key $n$ is fresh for $P$. The rules are slightly reformulated compared to [19] in that we use structural congruence $\equiv$. The rules for structural congruence are:

| | | |
|---|---|---|
| $P \mid 0 \equiv P$ | $P_0 \mid P_1 \equiv P_1 \mid P_0$ | $P_0 \mid (P_1 \mid P_2) \equiv (P_0 \mid P_1) \mid P_2$ |
| $P + 0 \equiv P$ | $P_0 + P_1 \equiv P_1 + P_0$ | $P_0 + (P_1 + P_2) \equiv (P_0 + P_1) + P_2$ |

**Table 1.** Forwards semantics of CCSK [17]

$$\frac{\mathsf{std}(P)}{\alpha.P \xrightarrow{\alpha[n]} \alpha[n].P} \qquad \frac{P \xrightarrow{\mu[m]} P' \quad m \neq n}{\alpha[n].P \xrightarrow{\mu[m]} \alpha[n].P'} \qquad \frac{P \equiv Q \xrightarrow{\mu[n]} Q' \equiv P'}{P \xrightarrow{\mu[n]} P'}$$

$$\frac{P_0 \xrightarrow{\mu[n]} P_0' \quad \mathsf{fsh}[n](P_1)}{P_0 \mid P_1 \xrightarrow{\mu[n]} P_0' \mid P_1} \qquad \frac{P_0 \xrightarrow{\alpha[n]} P_0' \quad P_1 \xrightarrow{\overline{\alpha}[n]} P_1'}{P_0 \mid P_1 \xrightarrow{\tau[n]} P_0' \mid P_1'}$$

$$\frac{P_0 \xrightarrow{\mu[n]} P_0' \quad \mathsf{std}(P_1)}{P_0 + P_1 \xrightarrow{\mu[n]} P_0' + P_1} \qquad \frac{P \xrightarrow{\mu[n]} P' \quad \mu, \overline{\mu} \notin A}{P \setminus A \xrightarrow{\mu[n]} P' \setminus A} \qquad \frac{P \xrightarrow{\mu[n]} P'}{P[f] \xrightarrow{f(\mu)[n]} P'[f]}$$

We extend CCSK with recursion as follows. We add process constants $A\left\langle \tilde{b} \right\rangle$, together with definitions $A(\tilde{a}) = P_A$, where $P_A$ is a standard process and $\tilde{a}$ is a tuple containing the actions of $P_A$. This leads us to expand our definition of structural congruence with $A\left\langle \tilde{b} \right\rangle \equiv P_A\{^{\tilde{b}}/_{\tilde{a}}\}$.

**Definition 2.1.** *A process $P$ is* reachable *if there exists a standard process $Q$ such that $Q(\rightarrow \cup \rightsquigarrow)^* P$, and* forwards-reachable *if there exists a standard process $Q$ such that $Q \rightarrow^* P$.*

Since CCSK is causal all reachable processes are forwards-reachable ( [19], Proposition 5.15; the proof still applies with recursion added).

## 3 Reversible Bundle Event Structures

Bundle event structures (BES) [11] extend prime event structures by allowing multiple possible causes for the same event. They do this by replacing the causal relation with a bundle set, so that if $X \mapsto e$ then exactly one of the events in $X$ must have happened before $e$ can happen, and all the events in $X$ must be in conflict.

We define reversible bundle event structures (RBES) by extending the bundle relation to map to reverse events, denoted $\underline{e}$, and adding a prevention relation, such that if $e \rhd \underline{e'}$ then $e'$ cannot be reversed from configurations containing $e$. We use $e^*$ to denote either $e$ or $\underline{e}$.

**Definition 3.1 (Reversible Bundle Event Structure).** *A reversible bundle event structure is a 5-tuple $\mathcal{E} = (E, F, \mapsto, \sharp, \rhd)$ where:*

1. *$E$ is the set of events;*
2. *$F \subseteq E$ is the set of reversible events;*
3. *the bundle set, $\mapsto \subseteq 2^E \times (E \cup \underline{F})$, satisfies $X \mapsto e^* \Rightarrow \forall e_1, e_2 \in X. e_1 \neq e_2 \Rightarrow e_1 \sharp e_2$ and for all $e \in F$, $\{e\} \mapsto \underline{e}$;*
4. *the conflict relation, $\sharp \subseteq E \times E$, is symmetric and irreflexive;*
5. *$\rhd \subseteq E \times \underline{F}$ is the prevention relation.*

In order to obtain a category of RBESs, we define a morphism in Definition 3.2.

**Definition 3.2 (RBES-morphism).** *Given RBESs $\mathcal{E}_0 = (E_0, F_0, \mapsto_0, \sharp_0, \rhd_0)$ and $\mathcal{E}_1 = (E_1, F_1, \mapsto_1, \sharp_1, \rhd_1)$, an RBES-morphism from $\mathcal{E}_0$ to $\mathcal{E}_1$ is a partial function $f : E_0 \to E_1$ such that $f(F_0) \subseteq F_1$ and for all $e, e' \in E_0$:*

1. *if $f(e) \sharp_1 f(e')$ then $e \sharp_0 e'$;*
2. *if $f(e) = f(e')$ and $e \neq e'$ then $e \sharp_0 e'$;*
3. *for $X_1 \subseteq E_1$ if $X_1 \mapsto_1 f(e)^*$ then there exists $X_0 \subseteq E_0$ such that $X_0 \mapsto_0 e^*$, $f(X_0) \subseteq X_1$, and if $e' \in X_0$ then $f(e') \neq \bot$;*
4. *if $f(e) \rhd_1 \underline{f(e')}$ then $e \rhd_0 \underline{e'}$.*

It can be checked that RBESs with this notion of morphism form a category **RBES**. We define a product of RBESs in Definition 3.3. A coproduct can also be defined similarly to other coproducts of event structures.

**Definition 3.3 (Product of RBESs).** *Let $\mathcal{E}_0 = (E_0, F_0, \mapsto_0, \sharp_0, \rhd_0)$ and $\mathcal{E}_1 = (E_1, F_1, \mapsto_1, \sharp_1, \rhd_1)$ be reversible bundle event structures. Their product $\mathcal{E}_0 \times \mathcal{E}_1$ is the RBES $\mathcal{E} = (E, F, \mapsto, \sharp, \rhd)$ with projections $\pi_0$ and $\pi_1$ where:*

1. *$E = E_0 \times_* E_1 = \{(e, *) \mid e \in E_0\} \cup \{(*, e) \mid e \in E_1\} \cup \{(e, e') \mid e \in E_0 \text{ and } e' \in E_1\}$;*
2. *$F = F_0 \times_* F_1 = \{(e, *) \mid e \in F_0\} \cup \{(*, e) \mid e \in F_1\} \cup \{(e, e') \mid e \in F_0 \text{ and } e' \in F_1\}$;*
3. *for $i \in \{0, 1\}$ we have $(e_0, e_1) \in E$, $\pi_i((e_0, e_1)) = e_i$;*
4. *for any $e^* \in E \cup \underline{F}$, $X \subseteq E$, $X \mapsto e^*$ iff there exists $i \in \{0, 1\}$ and $X_i \subseteq E_i$ such that $X_i \mapsto \pi_i(e)^*$ and $X = \{e' \in E \mid \pi_i(e') \in X_i\}$;*
5. *for any $e, e' \in E$, $e \sharp e'$ iff there exists $i \in \{0, 1\}$ such that $\pi_i(e) \sharp_i \pi_i(e')$, or $\pi_i(e) = \pi_i(e') \neq \bot$ and $\pi_{1-i}(e) \neq \pi_{1-i}(e')$;*
6. *for any $e \in E$, $e' \in F$, $e \rhd \underline{e'}$ iff there exists $i \in \{0, 1\}$ such that $\pi_i(e) \rhd_i \underline{\pi_i(e')}$.*

We wish to model RBESs as configuration systems (CSs), and therefore define a functor from one category to the other in Definition 3.5. A CS consists of a set of events, some of which are reversible, configurations of these events, and labelled transitions between them, as described in Definition 3.4. We will later use the CSs corresponding to our event structure semantics to describe the operational correspondence between our event structure semantics and the operational semantics of CCSK.

**Definition 3.4 (Configuration system [16]).** *A* configuration system *(CS) is a quadruple $\mathcal{C} = (E, F, \mathsf{C}, \to)$ where $E$ is a set of events, $F \subseteq E$ is a set of reversible events, $\mathsf{C} \subseteq 2^E$ is the set of configurations, and $\to \subseteq \mathsf{C} \times 2^{E \cup \underline{F}} \times \mathsf{C}$ is a labelled transition relation such that if $X \xrightarrow{A \cup \underline{B}} Y$ then:*

- *$X, Y \in \mathsf{C}$, $A \cap X = \emptyset$; $B \subseteq X \cap F$; and $Y = (X \setminus B) \cup A$;*
- *for all $A' \subseteq A$ and $B' \subseteq B$, we have $X \xrightarrow{A' \cup \underline{B'}} Z \xrightarrow{(A \setminus A') \cup \underline{(B \setminus B')}} Y$, meaning $Z = (X \setminus B') \cup A' \in \mathsf{C}$.*

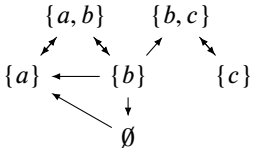
**Definition 3.5 (From RBES to CS).** *The functor $C_{br}$ : **RBES** $\to$ **CS** is defined as:*

1. *$C_{br}((E, F, \mapsto, \sharp, \rhd)) = (E, F, \mathsf{C}, \to)$ where:*
   *(a) $X \in \mathsf{C}$ if $X$ is conflict-free;*

(b) *For $X, Y \in \mathsf{C}$, $A \subseteq E$, and $B \subseteq F$, there exists a transition $X \xrightarrow{A \cup \underline{B}} Y$ if:*
   i. *$Y = (X \setminus B) \cup A$; $X \cap A = \emptyset$; $B \subseteq X$; and $X \cup A$ conflict-free;*
   ii. *for all $e \in B$, if $e' \rhd \underline{e}$ then $e' \notin X \cup A$;*
   iii. *for all $e \in A$ and $X' \subseteq E$, if $X' \mapsto e$ then $X' \cap (X \setminus B) \neq \emptyset$;*
   iv. *for all $e \in B$ and $X' \subseteq E$, if $X' \mapsto \underline{e}$ then $X' \cap (X \setminus (B \setminus \{e\})) \neq \emptyset$.*

2. *$C_{br}(f) = f$.*

Example 3.6 shows an RBES mapped to a CS. The configuration $\{b, c\}$ is reachable despite $b$ being required for $c$ to happen and $c$ being a possible cause of $b$.




*Example 3.6 (RBES).* An RBES $\mathcal{E} = (E, F, \mapsto, \sharp, \rhd)$ where $E = \{a, b, c\}$, $F = \{a, b\}$, $a \sharp c$, $\{a, c\} \mapsto b$, $\{b\} \mapsto c$ $\{a\} \mapsto \underline{a}$, $\{b\} \mapsto \underline{a}$, and $\{b\} \mapsto \underline{b}$, gives the CS $C_{br}(\mathcal{E})$.

We define a causal variant of RBESs in Definition 3.7. The subcategory **CRBES** consists of CRBESs and the RBES-morphisms between them.

**Definition 3.7 (Causal RBES).** *$\mathcal{E} = (E, F, \mapsto, \sharp, \rhd)$ is a causal RBES (CRBES) if (1) if $e \rhd \underline{e'}$ then either $e \sharp e'$ or there exists an $X \subseteq E$ such that $X \mapsto e$ and $e' \in X$, (2) if $X \mapsto e$ and $e' \in X \cap F$, then $e \rhd \underline{e'}$, and (3) if $X \mapsto \underline{e}$ then $e \in X$.*

**Proposition 3.8.**

1. *Given a CRBES, $\mathcal{E} = (E, F, \mapsto, \sharp, \rhd)$ and corresponding CS $C_{rb}(\mathcal{E}) = (E, F, \mathsf{C}, \rightarrow)$, any reachable $X \in \mathsf{C}$ is forwards-reachable.*
2. *If $\mathcal{E} = (E, F, \mapsto, \sharp, \rhd)$ is a CRBES and $C_{br}(\mathcal{E}) = (E, F, \mathsf{C}, \rightarrow)$ then whenever $X \in \mathsf{C}$, $X \xrightarrow{A \cup \underline{B}} Y$ and $A \cup B \subseteq F$, we get a transition $Y \xrightarrow{B \cup \underline{A}} X$.*

Since our motivation for defining RBESs was modelling reversible processes, we need to be able to label our events with a corresponding action from a process. For this we use a *labelled RBES* (LRBES).

**Definition 3.9 (Labelled Reversible Bundle Event Structure).** *An LRBES $\mathcal{E} = (E, F, \mapsto, \sharp, \rhd, \lambda, \mathsf{Act})$ consists of an RBES $(E, F, \mapsto, \sharp, \rhd)$, a set of labels $\mathsf{Act}$, and a surjective labelling function $\lambda : E \to \mathsf{Act}$.*

**Definition 3.10 (LRBES-morphism).** *Given LRBESs $\mathcal{E}_0 = (E_0, F_0, \mapsto_0, \sharp_0, \rhd_0, \lambda_0, \mathsf{Act}_0)$ and $\mathcal{E}_1 = (E_1, F_1, \mapsto_1, \sharp_1, \rhd_1, \lambda_1, \mathsf{Act}_1)$, an* LRBES-morphism *$f : \mathcal{E}_0 \to \mathcal{E}_1$ is a partial function $f : E_0 \to E_1$ such that $f : (E_0, F_0, \mapsto_0, \sharp_0, \rhd_0) \to (E_1, F_1, \mapsto_1, \sharp_1, \rhd_1)$ is an RBES-morphism and for all $e \in E_0$, either $f(e) = \bot$ or $\lambda_0(e) = \lambda_1(f(e))$.*

## 4 Event Structure Semantics of CCSK

Having defined RBESs, we will now use them to describe the semantics of CCSK [19]. Unlike the event structure semantics of CCS [2, 21], our semantics will generate both an event structure and an *initial configuration* containing all the events corresponding to past actions. This means that if $P \to P'$ then $P$ and $P'$ will be described by the same event structure with different initial states.

First we define the operators we will use in the semantics, particularly restriction, parallel composition, choice, and action prefixes. Restriction is achieved by simply removing any events associated with the restricted action.

**Definition 4.1 (Restriction).** *Given an LRBES,* $\mathcal{E} = (E, F, \mapsto, \sharp, \rhd, \lambda, \mathsf{Act})$*, restricting* $\mathcal{E}$ *to* $E' \subseteq E$ *creates* $\mathcal{E} \upharpoonright E' = (E', F', \mapsto', \sharp', \rhd', \lambda', \mathsf{Act}')$ *where:*

1. $F' = F \cap E'$;
2. $\mapsto' = \mapsto \cap (\mathcal{P}(E') \times (E' \cup \underline{F'}))$;
3. $\sharp' = \sharp \cap (E' \times E')$;
4. $\rhd' = \rhd \cap (E' \times \underline{F'})$;
5. $\lambda' = \lambda \upharpoonright_{E'}$;
6. $\mathsf{Act}$ *is the range of* $\lambda$.

Parallel composition uses the product of RBESs, labels as $\tau$ any event corresponding to a synchronisation, and removes any invalid events describing an impossible synchronisation.

**Definition 4.2 (Parallel).** *Given LRBESs* $\mathcal{E}_0$ *and* $\mathcal{E}_1$, $\mathcal{E}_0 || \mathcal{E}_1 = (E, F, \mapsto, \sharp, \rhd, \lambda, \mathsf{Act}) \upharpoonright \{e \mid \lambda(e) \neq 0\}$ *where:* $(E, F, \mapsto, \sharp, \rhd) = (E_0, F_0, \mapsto_0, \sharp_0, \rhd_0) \times (E_1, F_1, \mapsto_1, \sharp_1, \rhd_1)$;

$$\lambda(e) = \begin{cases} \lambda_0(e_0) & \text{if } e = (e_0, *) \\ \lambda_1(e_1) & \text{if } e = (*, e_1) \\ \tau & \text{if } e = (e_0, e_1) \text{ and } \lambda_0(e_0) = \overline{\lambda_1(e_1)} \\ 0 & \text{if } e = (e_0, e_1) \text{ and } \lambda_0(e_0) \neq \overline{\lambda_1(e_1)}; \end{cases}$$

*and* $\mathsf{Act} = \mathsf{Act}_0 \cup \mathsf{Act}_1 \cup \{0, \tau\}$.

Choice, which acts as a coproduct of LRBESs, simply uses the coproduct of RBESs, and defines the labels as expected.

**Definition 4.3 (Choice).** *Given LRBESs* $\mathcal{E}_0$ *and* $\mathcal{E}_1$, $\mathcal{E}_0 \& \mathcal{E}_1 = (E, F, \mapsto, \sharp, \rhd, \lambda, \mathsf{Act})$ *where:* $(E, F, \mapsto, \sharp, \rhd) = (E_0, F_0, \mapsto_0, \sharp_0, \rhd_0) + (E_1, F_1, \mapsto_1, \sharp_1, \rhd_1)$; $\lambda(i_j(e)) = \lambda_j(e)$; *and* $\mathsf{Act} = \mathsf{Act}_0 \cup \mathsf{Act}_1$.

Causally prefixing an action onto an event structure means the new event causes all other events and is prevented from reversing by all other events.

**Definition 4.4 (Causal Prefix).** *Given an LRBES* $\mathcal{E} = (E, F, \mapsto, \sharp, \rhd, \lambda, \mathsf{Act})$*, an event* $e \notin E$*, and a label* $\alpha$, $\alpha(e).\mathcal{E} = (E', F', \mapsto', \sharp', \rhd', \lambda', \mathsf{Act}')$ *where:*

1. $E' = E \cup \{e\}$;
2. $F' = F \cup \{e\}$;
3. $\mapsto' = \mapsto \cup (\{\{e\}\} \times (E \cup \{\underline{e}\}))$;
4. $\sharp' = \sharp$;
5. $\rhd' = \rhd \cup (E \times \{\underline{e}\})$;
6. $\lambda' = \lambda[e \mapsto \alpha]$;
7. $\mathsf{Act}' = \mathsf{Act} \cup \{\alpha\}$.

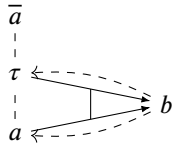
Now that we have defined the main operations of the process calculus, we define the event structure semantics in Table 2. We do this using rules of the form $[\![P]\!]_l = \langle \mathcal{E}, \mathsf{Init}, k \rangle$ wherein $l$ is the level of unfolding, which we use to model recursion, $\mathcal{E}$ is an LRBES, $\mathsf{Init}$ is the initial configuration, and $k : \mathsf{Init} \to \mathcal{K}$ is a function assigning communication keys to the past actions, which we use in parallel composition to determine which synchronisations of past actions to put in $\mathsf{Init}$.

**Table 2.** RBES-semantics of CCSK

| | |
|---|---|
| $\{\!\{0\}\!\}_l =$ | $\langle(\emptyset,\emptyset,\emptyset,\emptyset,\emptyset,\emptyset,\emptyset),\emptyset,\emptyset\rangle$ |
| $\{\!\{P_0 + P_1\}\!\}_l =$ | $\langle \mathcal{E}_0 \& \mathcal{E}_1, \mathsf{Init}, k\rangle$ where |
| | For $i \in \{0,1\}$, $\{\!\{P_i\}\!\}_l = \langle \mathcal{E}_i, \mathsf{Init}_i, k_i\rangle$ |
| | $\mathsf{Init} = \{(j,e) \mid j \in \{0,1\} \text{ and } e \in \mathsf{Init}_j\}$ |
| | $k((j,e)) = k_j(e)$ if $e \in \mathsf{Init}_j$ |
| $\{\!\{\alpha.P\}\!\}_l =$ | $\left\langle \alpha(e).(E, F, \mapsto, \sharp, \rhd, \lambda, \mathsf{Act}), \mathsf{Init}, k \right\rangle$ for $e$ fresh for $E$ where |
| | $\{\!\{P\}\!\}_l = \left\langle (E, F, \mapsto, \sharp, \rhd, \lambda, \mathsf{Act}), \mathsf{Init}, k \right\rangle$ |
| $\{\!\{\alpha[m].P\}\!\}_l =$ | $\left\langle \alpha(e).(E, F, \mapsto, \sharp, \rhd, \lambda, \mathsf{Act}), \mathsf{Init} \cup \{e\}, k[e \mapsto m] \right\rangle$ for $e$ fresh for $E$ where |
| | $\{\!\{P\}\!\}_l = \left\langle (E, F, \mapsto, \sharp, \rhd, \lambda, \mathsf{Act}), \mathsf{Init}, k \right\rangle$ |
| $\{\!\{P_0 \mid P_1\}\!\}_l =$ | $\left\langle (E, F, \mapsto, \sharp, \rhd, \lambda, \mathsf{Act}), \mathsf{Init}, k \right\rangle$ where |
| | For $i \in \{0,1\}$, $\{\!\{P_i\}\!\}_l = \langle \mathcal{E}_i, \mathsf{Init}_i, k_i\rangle$ |
| | $(E, F, \mapsto, \sharp, \rhd, \lambda, \mathsf{Act}) = \mathcal{E}_0 \| \mathcal{E}_1$ |
| | $\mathsf{Init} = \{(e_0, e_1) \mid e_0 \in \mathsf{Init}_0, e_1 \in \mathsf{Init}_1, k_0(e_0) = k_1(e_1)\} \cup$ |
| | $\{(*, e_1) \mid e_1 \in \mathsf{Init}_1 \text{ and } \nexists e_0 \in \mathsf{Init}_0 . \lambda_0(e_0) = \overline{\lambda_1(e_1)} \text{ and } k_0(e_0) = k_1(e_1)\} \cup$ |
| | $\{(e_0, *) \mid e_0 \in \mathsf{Init}_0 \text{ and } \nexists e_1 \in \mathsf{Init}_1 . \lambda_0(e_0) = \overline{\lambda_1(e_1)} \text{ and } k_0(e_0) = k_1(e_1)\}$ |
| | $k(e) = \begin{cases} k_0(e_0) & \text{if } e = (e_0, *) \\ k_1(e_1) & \text{if } e = (*, e_1) \\ k_0(e_0) & \text{if } e = (e_0, e_1) \quad \textit{– note that } k_0(e_0) = k_1(e_1) \end{cases}$ |
| $\{\!\{P \setminus A\}\!\}_l =$ | $\langle \mathcal{E} \upharpoonright \{e \mid \lambda(e) \notin \mathcal{A}\}, \mathsf{Init} \cap \{e \mid \lambda(e) \notin \mathcal{A}\}, k \upharpoonright \{e \mid \lambda(e) \notin \mathcal{A}\}\rangle$ where |
| | $\{\!\{P\}\!\}_l = \langle \mathcal{E}, \mathsf{Init}, k\rangle$ |
| | $\mathcal{A} = A \cup \overline{A}$ |
| $\{\!\{P[f]\}\!\}_l =$ | $\left\langle (E, F, \mapsto, \sharp, \rhd, \lambda, \mathsf{Act}), \mathsf{Init}, k \right\rangle$ where |
| | $\{\!\{P\}\!\}_l = \left\langle (E, F, \mapsto, \sharp, \rhd, \lambda', \mathsf{Act}'), \mathsf{Init}, k \right\rangle$ |
| | $\mathsf{Act} = f(\mathsf{Act}')$ |
| | $\lambda = f \circ \lambda'$ |
| $\left\{\!\!\left\{ A\left\langle \tilde{b} \right\rangle \right\}\!\!\right\}_0 =$ | $\langle(\emptyset,\emptyset,\emptyset,\emptyset,\emptyset,\emptyset,\emptyset),\emptyset,\emptyset\rangle$ |
| $\left\{\!\!\left\{ A\left\langle \tilde{b} \right\rangle \right\}\!\!\right\}_l =$ | $\left\{\!\!\left\{ P_A\{^{\tilde{b}}/_{\tilde{a}}\} \right\}\!\!\right\}_{l-1}$ where $A(\tilde{a}) = P_A$ |

Note that the only difference between a future and a past action is that the event corresponding to a past action is put in the initial state and given a communication key.

*Example 4.5.* The CCSK process $a.b \mid \overline{a}$ (cf. Example 1.1) can be represented by the RBES with events labelled $a$, $\overline{a}$, $\tau$, and $b$, the bundle $\{a, \tau\} \mapsto b$, the conflicts $a \sharp \tau$ and $\overline{a} \sharp \tau$, and the preventions $b \rhd \underline{a}$ and $b \rhd \underline{\tau}$.




We say that $\{\!\{P\}\!\} = \sup_{l \in \mathcal{N}} \{\!\{P\}\!\}_l$. This means we need to show that there exists such a least upper bound of the levels of unfolding. As shown in [8], ordering closed BESs by restriction produces a complete partial order. Since our LRBESs do not have overlapping bundles ($X \mapsto e^*$ and $X' \mapsto e^*$ implies $X \neq X'$ or $X \cap X' = \emptyset$) they are closed, and we can use a similar ordering.

**Definition 4.6 (Ordering of LRBESs).** *Given LRBESs* $\mathcal{E}_0 = (E_0, F_0, \mapsto_0, \sharp_0, \rhd_0, \lambda_0, \mathsf{Act}_0)$ *and* $\mathcal{E}_1 = (E_1, F_1, \mapsto_1, \sharp_1, \rhd_1, \lambda_1, \mathsf{Act}_1)$*,* $\mathcal{E}_0 \leq \mathcal{E}_1$ *if* $\mathcal{E}_0 = \mathcal{E}_1 \upharpoonright E_0$*.*

**Proposition 4.7 (Unfolding).** *Given a reachable process P and a level of unfolding l, if* $\{\!\{P\}\!\}_l = \langle \mathcal{E}, \mathsf{Init}, k\rangle$ *and* $\{\!\{P\}\!\}_{l-1} = \left\langle \mathcal{E}', \mathsf{Init}', k' \right\rangle$*, then* $\mathcal{E}' \leq \mathcal{E}$*,* $\mathsf{Init} = \mathsf{Init}'$*, and* $k = k'$*.*

In order to prove that our event structure semantics correspond with the operational semantics for CCSK defined in [17] we first show that our event structures are causal.

**Proposition 4.8.** *Given a process $P$ such that $\{\!|P|\!\} = \langle \mathcal{E}, \mathsf{Init}, k \rangle$, $\mathcal{E}$ is causal.*

Structurally congruent processes will generate isomorphic event structures:

**Proposition 4.9 (Structural Congruence).** *Given processes $P$ and $P'$, if $P \equiv P'$, $\{\!|P|\!\} = \langle \mathcal{E}, \mathsf{Init}, k \rangle$, and $\{\!|P'|\!\} = \left\langle \mathcal{E}', \mathsf{Init}', k' \right\rangle$, then there exists an isomorphism $f : \mathcal{E} \to \mathcal{E}'$ such that $f(\mathsf{Init}) = \mathsf{Init}'$ and for all $e \in \mathsf{Init}$, $k(e) = k'(f(e))$.*

Finally we show in Theorem 4.10 that given a process $P$ with a conflict-free initial state, including any reachable process, there exists a transition $P \xrightarrow{\mu} P'$ if and only if the event structure corresponding to $P$ is isomorphic to the event structure corresponding to $P'$ and an event $e$ labelled $\mu$ exists such that $e$ is available in $P$'s initial state, and $P'$'s initial state is $P$'s initial state with $e$ added.

**Theorem 4.10.** *Let $P$ be a process with $\{\!|P|\!\} = \langle \mathcal{E}, \mathsf{Init}, k \rangle$, $\mathcal{E} = (E, F, \mapsto, \sharp, \rhd, \lambda, \mathsf{Act})$, $C_{br}(\mathcal{E}) = (E, F, \mathsf{C}, \to)$, and $\mathsf{Init}$ conflict-free. Then*

1. *if there exists a $P'$ with $\{\!|P'|\!\} = \left\langle \mathcal{E}', \mathsf{Init}', k' \right\rangle$ and a transition $P \xrightarrow{\mu[m]} P'$ then there exists a transition $\mathsf{Init} \xrightarrow{\{e\}} X$ and an isomorphism $f : \mathcal{E} \to \mathcal{E}'$ such that $\lambda(e) = \mu$, $f \circ k' = k[e \mapsto m]$, and $f(X) = \mathsf{Init}'$;*
2. *and if there exists a transition $\mathsf{Init} \xrightarrow{\{e\}} X$ then there exists a $P'$ with $\{\!|P'|\!\} = \left\langle \mathcal{E}', \mathsf{Init}', k' \right\rangle$ and a transition $P \xrightarrow{\mu[m]} P'$ and an isomorphism $f : \mathcal{E} \to \mathcal{E}'$ such that $\lambda(e) = \mu$, $f \circ k' = k[e \mapsto m]$, and $f(X) = \mathsf{Init}'$.*

**Corollary 4.11.** *Let $P$ be a process such that $\{\!|P|\!\} = \langle \mathcal{E}, \mathsf{Init}, k \rangle$. Then $\mathsf{Init}$ is forwards-reachable in $\mathcal{E}$ if and only if there exists a standard process $Q$ such that $Q \to^* P$.*

Since we showed in Proposition 4.8 that any event structures generated by processes are causal, it follows that we get a similar correspondence between the reverse transitions of processes and event structures.

## 5 Reversible Extended Bundle Event Structures

In CCSK a process can reverse actions at any time. Suppose that we wish to control this reversibility by having a 'rollback' action that causes all actions, or all actions since the last safe state, to be reversed before the process can continue, similar to the $\mathsf{roll}$ command of [9]. RBESs can easily ensure that this rollback event $\mathsf{roll}$ is required for other events to reverse; we simply say that $\{\mathsf{roll}\} \mapsto \underline{e}$ for all $e$. However, preventing events from happening during the roll in RBESs requires symmetric conflict, which would mean the other events also prevent $\mathsf{roll}$ from occurring. To solve a similar problem, Phillips and Ulidowski [16] use reversible asymmetric event structures, which replace symmetric conflict with asymmetric. But since these use the same notion of causality

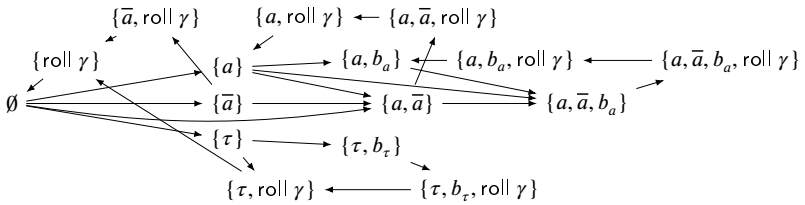



**Fig. 1.** The reachable configurations of the REBES described in Example 5.1

as reversible prime event structures, they have trouble modelling concurrent processes with synchronisation, as shown in Example 1.1.

Extended bundle event structures (EBES) [11] add asymmetric conflict; so defining a reversible variant of these will allow us to model the above scenario.

*Example 5.1 (The necessity of REBESs for modelling rollback).* Consider $a.b \mid \overline{a}_\gamma.\mathsf{roll}\ \gamma$, where $\mathsf{roll}\ \gamma$ means undo the action labelled $\gamma$, that is $\overline{a}$, and everything caused by it before continuing. To model this we would need to expand the RBES from Example 4.5 with a new event $\mathsf{roll}\ \gamma$, and split $b$ into two different events depending on whether it needs to be reversed during the rollback or not. This would give us an RBES $(\{a, \tau, \overline{a}, b_a, b_\tau, \mathsf{roll}\ \gamma\}, \{a, \tau, \overline{a}, b_a, b_\tau, \mathsf{roll}\ \gamma\}, \mapsto, \sharp, \rhd)$ where $\{a\} \mapsto b_a$, $\{\tau\} \mapsto b_\tau$, $\{\overline{a}, \tau\} \mapsto \mathsf{roll}\ \gamma$, $\{\mathsf{roll}\ \gamma\} \mapsto \underline{\tau}$, $\{\mathsf{roll}\ \gamma\} \mapsto \underline{\overline{a}}$, $\{\mathsf{roll}\ \gamma\} \mapsto \underline{b_\tau}$, $a \sharp \tau$, $\overline{a} \sharp \tau$, $b_a \rhd \underline{a}$, $b_\tau \rhd \underline{\tau}$, $\overline{a} \rhd \underline{\mathsf{roll}\ \gamma}$, and $\tau \rhd \underline{\mathsf{roll}\ \gamma}$. This would indeed ensure that $\overline{a}$ and the events caused by it could only reverse if one of the roll events had occurred, but it would not force them to do so before doing anything else. For this we use asymmetric conflict: $\mathsf{roll}\ \gamma \rhd \overline{a}$, $\mathsf{roll}\ \gamma \rhd \tau$, $\mathsf{roll}\ \gamma \rhd b_\tau$, giving us a CS with the *reachable configurations* shown in Figure 1.

We define a reversible version of EBESs in Definition 5.2, treating the asymmetric conflict similarly to RAESs in [16].

**Definition 5.2 (Reversible Extended Bundle Event Structure).** *An REBES is a 4-tuple $\mathcal{E} = (E, F, \mapsto, \rhd)$ where:*

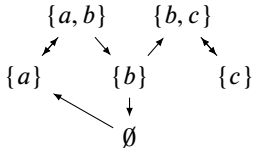
1. *$E$ is the set of events;*
2. *$F \subseteq E$ is the set of reversible events;*
3. *$\mapsto \subseteq 2^E \times (E \cup \underline{F})$ is the bundle set, satisfying $X \mapsto e \Rightarrow \forall e_1, e_2 \in X.(e_1 \neq e_2 \Rightarrow e_1 \rhd e_2)$, and for all $e \in F$, $\{e\} \mapsto \underline{e}$;*
4. *$\rhd \subseteq E \times (E \cup \underline{F})$ is the asymmetric conflict relation, which is irreflexive.*

In order to define REBES-morphisms, we extend the RBES morphism in the obvious way, letting the condition on preventions also apply to prevention on forwards events. This gives us a category **REBES**, in which we can define products and coproducts much like we did for RBESs, treating asymmetric conflict the same as we did symmetric.

We again model REBESs as CSs, defining configurations as sets of events on which $\rhd$ is well-founded, and extending the requirements of prevention in transitions to forwards events.

Example 5.3 shows an REBES, which cannot be represented by an RBES, since we get a transition $\emptyset \to \{a\}$, but no $\{b\} \to \{a, b\}$, despite $\{a, b\}$ being a configuration.

*Example 5.3 (REBES).* An REBES $\mathcal{E} = (E, F, \mapsto, \rhd)$ where $E = \{a, b, c\}$, $F = \{a, b\}, \{a, c\} \mapsto b$, $\{b\} \mapsto c$ $\{a\} \mapsto \underline{a}$, $\{b\} \mapsto \underline{a}$, $\{b\} \mapsto \underline{b}$, $a \rhd c$, $c \rhd a$, and $b \rhd a$ gives the CS $C_{er}(\mathcal{E})$ in the diagram.




Since we are using our REBESs for modelling the semantics of rollback in CCSK, we need a labelled variant, which we can define much as we did labelled RBESs.

## 6 Roll-CCSK

The operational semantics for roll-$\pi$ [9] are not translatable directly to CCSK, as they make use of the fact that one can know, when looking at a memory, whether the communication it was associated with was with another process or not, and therefore, for a given subprocess $P$ and a memory $m$, one knows whether all the memories and subprocesses caused by $m$ are part of $P$. In CCSK, this is not as easy, as the roll in a subprocess $\alpha_\gamma[n] \ldots \mathsf{roll}\ \gamma$, where $\gamma$ is a *tag* denoting which rollback rolls back which action, may or may not require rolling back the other end of the $\alpha$ communication, and all actions caused by it. We therefore need to check at every instance of parallel composition whether any communication has taken place, and if so roll back those actions and all actions caused by them. This may include rolling back additional actions from the subprocess containing the roll as in $a[n_1].\overline{b}[n_2] \mid c[n_3].(\overline{a}_\gamma[n_1].\mathsf{roll}\ \gamma \mid b[n_2])$, where it does not become clear that $b[n_2]$ needs to be reversed during the roll until the outer parallel composition. Unlike [9], we therefore do not provide low-level operational semantics for Roll-CCSK, only providing high-level operational semantics in this section, and low-level denotational event structure semantics in Section 7.

The syntax of Roll-CCSK is as follows:

$$P ::= \alpha_\gamma.P \mid \alpha_\gamma[n].P \mid P_0 + P_1 \mid P_0 | P_1 \mid P\backslash A \mid P[f] \mid 0 \mid \mathsf{roll}\ \gamma \mid \mathsf{rolling}\ \gamma \mid (\nu\, \gamma)P$$

Most of the syntax is the same as CCSK and CCS, but adding tags and rolls as described above, and $\mathsf{rolling}\ \gamma$, which denotes a roll in progress, the necessity of which is justified later. From now on we will use $\alpha.P$ to denote $\alpha_\gamma.P$ where no $\mathsf{roll}\ \gamma$ exists in $P$. Before presenting the operational semantics of rollback, we define causal dependence and projection similarly to [9], on which we base our own semantics.

**Definition 6.1 (Causal dependence).** *Let $P$ be a process and $\Gamma$ be the set of tags in $P$. Then the binary relation $\leq_P$ is the smallest relation satisfying*

- *if there exists a process $P'$ and past actions $\alpha_\gamma[n]$ and $\beta_{\gamma'}[m]$ such that $\alpha_\gamma[n].P'$ is a subprocess of $P$ and $\beta_{\gamma'}[m]$ occurs in $P'$ then $\gamma \leq_P \gamma'$;*
- *if there exist past actions $\alpha_\gamma[n]$ and $\beta_{\gamma'}[n]$ in $P$ with the same keys then $\gamma \leq_P \gamma'$;*
- *$\leq_P$ is reflexively and transitively closed.*

**Table 3.** The main rules for rollback in the operational semantics of Roll-CCSK

| | | | |
|---|---|---|---|
| (start ROLL) | $\mathsf{roll}\ \gamma \overset{\text{start roll } \gamma}{\rightsquigarrow} \mathsf{rolling}\ \gamma$ | (par ROLL) | $\dfrac{P_0 \overset{\text{roll } \gamma}{\rightsquigarrow} P_0' \quad C = \{\gamma' \mid \gamma \leq_{P_0 \mid P_1} \gamma'\}}{P_0 \mid P_1 \overset{\text{roll } \gamma}{\rightsquigarrow} (P_0 \mid P_1)_{\natural C}}$ |
| (ROLL) | $\mathsf{rolling}\ \gamma \overset{\text{roll } \gamma}{\rightsquigarrow} \mathsf{roll}\ \gamma$ | (act ROLL) | $\dfrac{P \overset{\text{roll } \gamma}{\rightsquigarrow} P' \quad C = \{\gamma' \mid \gamma \leq_{\alpha_\gamma[n].P} \gamma'\}}{\alpha_\gamma[n].P \overset{\text{roll } \gamma}{\rightsquigarrow} \alpha_\gamma.P_{\natural C}}$ |
| (bind ROLL) | $\dfrac{P \overset{\text{roll } \gamma}{\rightsquigarrow} P'}{(\nu\ \gamma)P \overset{\text{roll bound}}{\rightsquigarrow} (\nu\ \gamma)P'}$ | (bind ROLL struct) | $\dfrac{P \equiv Q \overset{\text{roll bound}}{\rightsquigarrow} Q' \equiv P'}{P \overset{\text{roll bound}}{\rightsquigarrow} P'}$ |

**Definition 6.2 (Projection).** *Given a process $P$ and a set of tags $C$, $P_{\natural C}$ is defined as:*

$(\alpha_\gamma[n].P)_{\natural C} = \alpha_\gamma[n].(P_{\natural C})$ *if* $\gamma \notin C$ $\quad 0_{\natural C} = 0$ $\quad (P \setminus A)_{\natural C} = (P_{\natural C}) \setminus A$

$(\alpha_\gamma[n].P)_{\natural C} = \alpha_\gamma.(P_{\natural C})$ *if* $\gamma \in C$ $\quad \mathsf{roll}\ \gamma_{\natural C} = \mathsf{roll}\ \gamma$ $\quad (P_0 \mid P_1)_{\natural C} = P_{0_{\natural C}} \mid P_{1_{\natural C}}$

$\mathsf{rolling}\ \gamma_{\natural C} = \mathsf{rolling}\ \gamma$ *if* $\gamma \notin C$ $\quad A\left\langle \tilde{b}, \tilde{\gamma} \right\rangle_{\natural C} = A\left\langle \tilde{b}, \tilde{\gamma} \right\rangle$ $\quad (\nu\ \gamma)P_{\natural C} = (\nu\ \gamma)(P_{\natural C})$

$\mathsf{rolling}\ \gamma_{\natural C} = \mathsf{roll}\ \gamma$ *if* $\gamma \in C$ $\quad (\alpha_\gamma.P)_{\natural C} = \alpha_\gamma.(P_{\natural C})$ $\quad (P[f])_{\natural C} = P_{\natural C}[f]$

$(P_0 + P_1)_{\natural C} = P_{0_{\natural C}} + P_{1_{\natural C}}$

Much as in [9] we perform our rollback in two steps, the first triggering the rollback, and the second actually performing the rollback, in order to ensure that we can start multiple rollbacks at the same time. For instance, in the process $(a_\gamma.(\overline{d}.0 \mid c.\mathsf{roll}\ \gamma) \mid b_{\gamma'}.(\overline{c} \mid d.\mathsf{roll}\ \gamma') \mid \overline{a} \mid \overline{b}) \setminus \{a, b, c, d\}$ we will otherwise never be able to roll all the way back to the beginning, as rolling back $a_\gamma$ will stop us from reaching $\mathsf{roll}\ \gamma'$ and vice versa.

Table 3 shows the most important rules for reversing actions in Roll-CCSK. The remaining rules permit the $\mathsf{roll\ start}\ \gamma$ and $\mathsf{roll}\ \gamma$ to propagate in the same way as actions in CCSK (and past tag bindings), with the exception that in the rule for choice, if one path has already triggered a roll, the other cannot trigger or perform a roll or a forwards action. The semantics of forwards actions are otherwise identical to CCSK, except again propagating past the tag bindings. By contrast, $\mathsf{roll\ bound}\ \gamma$ does not propagate. We extend our process definitions $A(\tilde{a}) = P_A$ to also include a tuple of tags in $P_A$, giving us $A(\tilde{a}, \tilde{\gamma}) = P_A$, where $P_A$ is a standard process containing no instances of $\mathsf{rolling}\ \gamma$.

Since we want to be able to handle recursion without confusing instances of multiple actions or rollbacks being associated with the same tags, we introduce binding of tags $(\nu\ \gamma)$, which allows us to avoid clashes. We use $\mathsf{ft}(P)$ to denote the free tags of $P$. To ensure that we cannot perform $\mathsf{roll}\ \gamma$ in $Q \mid (\nu\ \gamma)P$ without rolling back all actions in $Q$ caused by $\gamma$, we only have rule (bind ROLL struct) for bound tags, meaning that to roll back a bound tag we must use structural congruence to move it to the outermost layer of the process. This is also why we have the two rules allowing us to move $(\nu\ \gamma)$ from one side of an action with a different tag to the other.

We also change the rule for applying definitions to ensure all tags are fresh for the unfolded process. This is again to prevent the process from unfolding more rollbacks for

a previous action, such as in $a_\gamma.A\langle a,\gamma\rangle$ with $A(b,\delta) = b_\delta.(A\,\langle b,\delta\rangle \mid \mathsf{roll}\ \delta)$, where there would otherwise be confusion about how far back one should roll each time.

Structural congruence for bound tags:

$\alpha_\gamma(\nu\gamma')P \equiv (\nu\gamma')\alpha_\gamma P$ if $\gamma \neq \gamma'$ $\quad$ $\alpha_\gamma[n](\nu\gamma')P \equiv (\nu\gamma')\alpha_\gamma[n]P$ if $\gamma \neq \gamma'$

$((\nu\gamma')P) \mid Q \equiv (\nu\gamma')(P \mid Q)$ if $\gamma \notin \mathsf{ft}(Q)$ $\quad$ $((\nu\gamma')P) + Q \equiv (\nu\gamma')(P + Q)$ if $\gamma \notin \mathsf{ft}(Q)$

$(\nu\gamma')(P \setminus A) \equiv ((\nu\gamma')P) \setminus A$ $\quad$ $(\nu\gamma)(P[f]) \equiv ((\nu\gamma)P)[f]$

$A\left\langle \tilde{b},\tilde{\delta}\right\rangle \equiv (\nu\ \tilde{\delta})P_A\{^{\tilde{b},\tilde{\delta}}/_{\tilde{a},\tilde{\gamma}}\}$ if $A(\tilde{a},\tilde{\gamma}) = P_A$ $\quad$ $(\nu\ \gamma)(\nu\gamma')P \equiv (\nu\ \gamma')(\nu\ \gamma)P$

*Example 6.3 (Bound Tags).* Consider the process $P = a_\gamma[n].(\nu\ \gamma)b_\gamma.\mathsf{roll}\ \gamma$. This can clearly do the actions $P \xrightarrow{b[m]} a_\gamma[n].(\nu\ \gamma)b_\gamma[m].\mathsf{roll}\ \gamma \xrightarrow{\text{start roll } \gamma} a_\gamma[n](\nu\ \gamma).b_\gamma[m].\mathsf{rolling}\ \gamma$. However, when actually performing the rollback, we need to use the structural congruence rule to $\alpha$-convert the bound $\gamma$ into $\delta$ and move the binding to before $a_\gamma[n]$ because roll bound does not propagate through $a_\gamma[n]$. Then we can do $a_\gamma[n].(\nu\ \gamma)b_\gamma[m].\mathsf{rolling}\ \gamma \equiv (\nu\ \delta)a_\gamma[n].b_\delta[m].\mathsf{rolling}\ \delta \xrightsquigarrow{\text{roll bound}} (\nu\ \delta)a_\gamma[n].b_\delta.\mathsf{roll}\ \delta$.

In addition, to ensure every rollback is associated with exactly one action, we define a *consistent process*.

**Definition 6.4 (Consistent process).** *A Roll-CCSK process $P$ is* consistent *if*

1. *there exists a standard process $Q$ with no subprocess* rolling $\gamma$ *such that $Q \to^* P$;*
2. *there exists $P' \equiv_\alpha P$, such that*
   (a) *for any tag $\gamma$, $P'$ has at most one subprocess* roll $\gamma$ *or* rolling $\gamma$*;*
   (b) *for any tag $\gamma$, there exists exactly one $\alpha$ and at most one $n$ such that $\alpha_\gamma$ or $\alpha_\gamma[n]$ occur in $P'$;*
   (c) *if* roll $\gamma$ *is a subprocess of $P'$ then there exists an action $\alpha$ and process $P''$ such that* roll $\gamma$ *is a subprocess of $P''$ and either $\alpha_\gamma.P''$ is a subprocess of $P'$ or there exists a key $n$ such that $\alpha_\gamma[n].P''$ is a subprocess of $P'$;*
3. *if $A\left\langle \tilde{b},\tilde{\delta}\right\rangle$ is a subprocess of $P$ defined as $A(\tilde{a},\tilde{\gamma}) = P_A$, then $P_A$ is consistent.*

**Proposition 6.5.** *Let $P$ be a consistent process, $P'$ be a process, and either $P \equiv P'$, $P \to P'$, or $P \rightsquigarrow P'$. Then $P'$ is consistent.*

We are then ready to prove Theorem 6.6, stating that for consistent subprocesses, any rollback can be undone by a sequence of forwards actions.

**Theorem 6.6 (Loop (Soundness)).** *Let $P_0$ and $P_1$ be consistent processes containing no subprocesses* rolling $\gamma$*, and such that $P_0 \xrightsquigarrow{\text{start roll } \gamma} P_0' \xrightsquigarrow{\text{roll bound}} P_1$. Then $P_1 \to^* P_0$.*

We will from now on use $\to_{CCSK}$ and $\rightsquigarrow_{CCSK}$ to distinguish CCSK-transitions from Roll-CCSK transitions, which will continue to be denoted by arrows without subscripts. The last thing we need to prove about our rollback operational semantics before moving on to event structure semantics is Theorem 6.9, stating that (1) our rollbacks only reverse the actions caused by the action we are rolling back according to CCSK, and (2) our rollbacks are *maximally permissive*, meaning that any subset of reached rollbacks may be successfully executed.

**Definition 6.7 (Transforming Roll-CCSK to CCSK).** *We define a function, $\phi$, which translates a Roll-CCSK process into CCSK:*

$\phi(\mathsf{roll}\ \gamma) = 0 \quad \phi(\alpha_\gamma[n].P) = \alpha[n].\phi(P) \quad \phi(\alpha_\gamma.P) = \alpha.\phi(P) \quad \phi((\nu\ \gamma)P) = \phi(P)$

*$\phi$ is otherwise homomorphic on the remainder.*

**Definition 6.8.** *Let $P$ be a CCSK process and $T = \{m_0, m_1, \ldots m_n\}$ be a set of keys. We say that $P \rightsquigarrow_T P'$ if there exist actions $\mu, \nu$ and a tag $m$ such that $P \overset{\mu[m]}{\rightsquigarrow}_{CCSK} P'$ and $\nu[m_i] \leq_P \mu[m]$ for some $m_i \in T$.*

**Theorem 6.9 (Completeness).** *Let $P$ be a consistent Roll-CCSK process with sub-processes $\alpha_{0_{\gamma_0}}[m_0] \ldots \mathsf{roll}\ \gamma_0$, $\alpha_{1_{\gamma_1}}[m_1] \ldots \mathsf{roll}\ \gamma_1$, $\ldots$, $\alpha_{n_{\gamma_n}}[m_n] \ldots \mathsf{roll}\ \gamma_n$. Then for all $T \subseteq \{m_0, m_1, \ldots m_n\}$, if $\phi(P) \rightsquigarrow_T^* P' \not\rightsquigarrow_T$ then there exists a Roll-CCSK process $P''$ such that $\phi(P'') = P'$ and $P \rightsquigarrow^* P''$.*

## 7 Event Structure semantics of Roll-CCSK

Having proved that our rollback semantics behave as intended, we are ready to translate them into event structure semantics in Table 4. We use labelled REBESs.

To model $\mathsf{roll}\ \gamma$ as an event structure, we have two events, one which triggers the roll, labelled $\mathsf{start\ roll}\ \gamma$, and another, $\mathsf{roll}\ \gamma$, which denotes that the roll is in progress, allowing the events caused by the associated action to begin reversing. When prefixing a process $P$ with an action $\alpha_\gamma$, we now need to ensure that any action in $P$, and any $\mathsf{start\ roll}$ associated with such an action, will be reversed by any $\mathsf{roll}\ \gamma$ in $P$, and that the rollback does not stop, signified by the event labelled $\mathsf{roll}\ \gamma$ being reversed, until those actions have all been reversed.

When composing the LREBESs of two processes, we also create a separate event for each set of causes it might have (Definition 7.1). This allows us to say that an event can be rolled back if it was caused by a communication with one of the events being rolled back, but not if the communication went differently. Consider the process $a_\gamma.\mathsf{roll}\ \gamma \mid \overline{a}.b \mid a_{\gamma'}.\mathsf{roll}\ \gamma'$. In this case we will want $b$ to roll back if $(a_\gamma, a)$ and $\mathsf{roll}\ \gamma$ have happened, or if $(a_{\gamma'}, a)$ and $\mathsf{roll}\ \gamma'$ have happened, but not if any other combination of the four events has happened, something which bundles cannot express unless $b$ is split into multiple events. In addition, we use the sets of causes to ensure that if $e$ is in $e'$'s set of causes and $e_{roll}$ can cause $e$ to reverse, then $e_{roll}$ can cause $e'$ to reverse.

**Definition 7.1.** *Given an LREBES, $\mathcal{E} = (E, F, \mapsto, \rhd, \lambda, \mathsf{Act})$, the set of possible causes for an event $e \in E$, $\mathsf{cause}(e) = X$, contains minimal sets of events such that if $x \in X$ then:*

1. *if $x' \mapsto e$ then there exists $e'$ such that $x' \cap x = \{e'\}$;*
2. *if $e' \in x$ then there exists $x' \in \mathsf{cause}(e')$ such that $x' \subseteq x$;*
3. *if $e_1, e_0 \in X$ then we cannot have both $e_0 \rhd e_1$ and $e_1 \rhd e_0$.*

When giving the semantics of restriction, we remove not only the actions associated with the restricted labels, but also the actions caused by them. This is because we want the event structures generated by $P$ and $0 \mid P$ always to be isomorphic; if $P = (a.b)\backslash\{a\}$,

we will otherwise get an event $b$, which, having no possible causes, disappears once we put $P$ in parallel with any other process, since this involves generating a $b$ event for each set of possible causes.

**Definition 7.2 (Removing labels and their dependants).** *Given an event structure $\mathcal{E} = (E, F, \mapsto, \rhd, \lambda, \mathsf{Act})$ and a set of labels $A \subseteq \mathsf{Act}$, we define $\rho(A) = X$ as the maximum subset of $E$ such that if $e \in X$ then $\lambda(e) \notin A$, and if $e \in X$ then there exists $x \in \mathsf{cause}(e)$ such that $x \subseteq X$.*

We give the REBES-semantics of Roll-CCSK in Table 4.

Much as we did in Proposition 4.7, we need to show that there exists a least upper bound of the event structures resulting from unfolding recursion. For this we first show that our action prefix, parallel composition, and tag binding are monotonic.

**Proposition 7.3 (Unfolding).** *Given a consistent process $P$ and a level of unfolding $l$, if $\{\!|P|\!\}_l = \langle \mathcal{E}, \mathsf{Init}, k\rangle$ and $\{\!|P|\!\}_{l-1} = \left\langle \mathcal{E}', \mathsf{Init}', k' \right\rangle$, then $\mathcal{E}' \leq \mathcal{E}$, $\mathsf{Init} = \mathsf{Init}'$, and $k = k'$.*

Structurally congruent processes result in isomorphic event structures:

**Proposition 7.4 (Structural Congruence).** *Given consistent Roll-CCSK-processes $P$ and $P'$, if $P \equiv P'$, $\{\!|P|\!\} = \langle \mathcal{E}, \mathsf{Init}, k\rangle$, and $\{\!|P'|\!\} = \left\langle \mathcal{E}', \mathsf{Init}', k' \right\rangle$, then there exists an isomorphism $f : \mathcal{E} \to \mathcal{E}'$ such that $f(\mathsf{Init}) = \mathsf{Init}'$ and for all $e \in \mathsf{Init}$, $k(e) = k'(f(e))$.*

Table 4: LREBES-semantics of Roll-CCSK

$\{\!|\mathsf{roll}\ \gamma|\!\}_l = \langle(\{e_r, e_t\}, \{e_r, e_t\}, \mapsto, \rhd, \lambda, \mathsf{Act}), \emptyset, \emptyset\rangle$ where:
$\{e_r\} \mapsto \underline{e_r}$, $\{e_t\} \mapsto \underline{e_t}$ $\quad \{e_t\} \mapsto e_r$, and $\{e_r\} \mapsto \underline{e_t}$ $\quad e_t \rhd \underline{e_r}$ and $e_r \rhd e_t$

$$\lambda(e) = \begin{cases} \mathsf{roll}\ \gamma & \text{if } e = e_r \\ \mathsf{start\ roll}\ \gamma & \text{if } e = e_t \end{cases} \qquad \mathsf{Act} = \{\mathsf{roll}\ \gamma, \mathsf{start\ roll}\ \gamma\}$$

$\{\!|\mathsf{rolling}\ \gamma|\!\}_l = \langle(\{e_r, e_t\}, \{e_r, e_t\}, \mapsto, \rhd, \lambda, \mathsf{Act}), \{e_t\}, \emptyset\rangle$ where:
$\{e_r\} \mapsto \underline{e_r}$, $\{e_t\} \mapsto \underline{e_t}$ $\quad \{e_t\} \mapsto e_r$, and $\{e_r\} \mapsto \underline{e_t}$ $\quad e_t \rhd \underline{e_r}$ and $e_r \rhd e_t$

$$\lambda(e) = \begin{cases} \mathsf{roll}\ \gamma & \text{if } e = e_r \\ \mathsf{start\ roll}\ \gamma & \text{if } e = e_t \end{cases} \qquad \mathsf{Act} = \{\mathsf{roll}\ \gamma, \mathsf{start\ roll}\ \gamma\}$$

$\{\!|\alpha_\gamma.P|\!\}_l = \langle(E, F, \mapsto, \rhd, \lambda, \mathsf{Act}), \mathsf{Init}, k\rangle$ where:
$\{\!|P|\!\} = \langle(E_P, F_P, \mapsto_P, \rhd_P, \lambda_P, \mathsf{Act}_P), \mathsf{Init}, k\rangle$
$E = E_P \cup \{e_\alpha\}$ where $e_\alpha$ fresh

$$E_{Roll} = \left\{ e \,\middle|\, \begin{array}{l} \lambda_P(e) \in \{\mathsf{roll}\ \gamma', \mathsf{roll\ bound}\} \text{ or} \\ \lambda_P(e) \in \{\mathsf{start\ roll}\ \gamma' \mid \nexists \beta, n.\beta_{\gamma'} \text{ or } \beta_{\gamma'}[n] \text{ occurs in } \alpha_\gamma.P\} \end{array} \right\}$$

$F = F_P \cup \{e_\alpha\}$
$X \mapsto e$ if $X \mapsto_P e$ or $X = \{e_\alpha\}$, $e \in E_P$, and $\lambda_P(e) \neq \mathsf{roll}\ \gamma'$
$X \mapsto \underline{e}$ if $X = \{e\}$, or $e = e_\alpha$ and $X = \{e' \mid \lambda_P(e') = \mathsf{roll}\ \gamma\}$, or
$e \in E_{Roll}$ and $X \mapsto_P \underline{e}$, or $e \notin E_{Roll}$, $\{e\} \neq X' \mapsto_P \underline{e}$, and $X = X' \cup \{e' \mid \lambda_P(e') = \mathsf{roll}\ \gamma\}$
$\rhd = \rhd_P \cup (E_{Roll} \times \{\underline{e_\alpha}\}) \cup (\{e_\alpha\} \times \{\underline{e_r} \mid \lambda_P(e_r) = \mathsf{roll}\ \gamma\}) \cup$
$(\{e_r \mid \lambda_P(e_r) = \mathsf{roll}\ \gamma\} \times (E_{Roll} \cup \{e_\alpha\}))$
$\mathsf{Act} = \mathsf{Act}_P \cup \{\alpha\}$

$$\text{For all } e \in E,\ \lambda(e) = \begin{cases} \lambda_P(e) & \text{if } e \in E_P \\ \alpha & \text{if } e = e_\alpha \end{cases}$$

Table 4: LREBES-semantics of Roll-CCSK (continued)

$\{\!|\alpha_\gamma[m].P|\!\}_l = \langle (E, F, \mapsto, \rhd, \lambda, \mathsf{Act}), \mathsf{Init}, k\rangle$ where:

$\{\!|\alpha_\gamma.P|\!\} = \left\langle (E, F, \mapsto, \rhd, \lambda, \mathsf{Act}), \mathsf{Init}', k' \right\rangle \quad \{e_\alpha\} = \{e \in E \mid \lambda(a) = \alpha \text{ and } \nexists X \subseteq E. X \mapsto e_\alpha\}$

$\mathsf{Init} = \mathsf{Init}' \cup \{e_\alpha\} \qquad k(e) = \begin{cases} k'(e) & \text{if } e \in \mathsf{Init}_P \\ m & \text{if } e = e_\alpha \end{cases}$

$\left\{\!\left| A\left\langle \tilde{b}, \tilde{\delta}\right\rangle \right|\!\right\}_0 = \langle (\emptyset, \emptyset, \emptyset, \emptyset, \emptyset, \emptyset, \emptyset, \emptyset), \emptyset, \emptyset\rangle$

$\left\{\!\left| A\left\langle \tilde{b}, \tilde{\delta}\right\rangle \right|\!\right\}_l = \left\{\!\left| (\nu\ \tilde{\delta}) P_A\{^{\tilde{b},\tilde{\delta}}/_{\tilde{a},\tilde{\gamma}}\} \right|\!\right\}_{l-1}$ where $A\langle \tilde{a}, \tilde{\gamma}\rangle = P_A$ and $l \geq 1$

$\{\!|P_0 \mid P_1|\!\}_l = \langle (E, F, \mapsto, \rhd, \lambda, \mathsf{Act}), \mathsf{Init}, k\rangle$ where:

$\{\!|P_i|\!\} = \mathcal{E}_i = (E_i, F_i, \mapsto_i, \rhd_i, \lambda_i, \mathsf{Act}_i)$ for $i \in \{0, 1\}$; $\mathcal{E}_0 || \mathcal{E}_1 = (E_\times, F_\times, \mapsto_\times, \rhd_\times, \lambda_\times, \mathsf{Act}_\times)$

$\mathsf{Init}_\times = \{(e_0, e_1) \mid e_0 \in \mathsf{Init}_0, e_1 \in \mathsf{Init}_1, k_0(e_0) = k_1(e_1)\} \cup$

$\{(*, e_1) \mid e_1 \in \mathsf{Init}_1, \nexists e_0 \in \mathsf{Init}_0. \lambda_0(e_0) = \overline{\lambda_1(e_1)}, \text{ and } k_0(e_0) = k_1(e_1)\} \cup$

$\{(e_0, *) \mid e_0 \in \mathsf{Init}_0, \nexists e_1 \in \mathsf{Init}_1. \lambda_0(e_0) = \overline{\lambda_1(e_1)}, \text{ and } k_0(e_0) = k_1(e_1)\}$

$E_{action} = \left\{ (X, e) \middle| \begin{array}{l} e \in E_\times, \lambda_\times(e) \notin \{\mathsf{roll}\ \gamma, \mathsf{roll\ bound}\}, \\ X \in \mathsf{cause}(e), \text{ and } \forall e' \in X. \exists X' \in \mathsf{cause}(e'). X' \subseteq X \end{array} \right\}$

$E_{roll} = \{e \mid e \in E_\times \text{ and } \lambda_\times(e) \in \{\mathsf{roll}\ \gamma, \mathsf{roll\ bound}\}\}$

$E = E_{action} \cup E_{roll}; \quad F_{action} = \{(X, e) \in E \mid e \in F_\times\}; \quad F_{roll} = E_{roll} \cap F_\times; \quad F = F_{action} \cup F_{roll}$

We define $\pi_0$ and $\pi_1$ such that for $(X, (e_0, e_1)) \in E_a$, $\pi_i((X, (e_0, e_1))) = e_i$, and

for $(e_0, e_1) \in E_r$, $\pi_i(e_0, e_1) = e_i$

$\{(X, e') \mid X' \subseteq X\} \mapsto (X', (e_0, e_1))$ if $e' \in X'$

$X \mapsto (e_0, e_1)$ if there exists $X'$ such that $X' \mapsto_\times e$ and $X = \{e' \mid (\pi_0(e'), \pi_1(e')) \in X'\}$

$X \mapsto \underline{e}$ if $X = \{e\}$ or $e = (X', e_\times)$ and $X = \bigcup \left\{ X'' \middle| \begin{array}{l} \exists i \in \{0,1\}, X_i \in E_i. X_i \mapsto \overline{\pi_i(e)} \\ \text{ or } \exists e_\times \in X'. X_i \mapsto \overline{\pi_i(e_\times)} \\ \text{, and } e' \in X'' \text{ iff } \pi_i(e') \in X_i \end{array} \right\}$ or

$e = (e_0, e_1)$ and there exists $X'$ such that $X' \mapsto_\times \underline{e}$ and $X = \{e' \mid (\pi_0(e'), \pi_1(e')) \in X'\}$

$e \rhd e'^*$ if there exists $i \in \{0, 1\}$ such that $\pi_i(e) \rhd_i \pi_i(e')^*$, or

$\pi_i(e) = \pi_i(e') \neq \bot$, and $e \neq e'$, $e'^* = e'$, or $e \neq e'$, and $e \in X \mapsto \underline{e'}$, or $e'^* = e'$ and $e, e' \in E_r$

$\mathsf{Act} = \mathsf{Act}_0 \cup \mathsf{Act}_1 \cup \{\tau\}$

$\lambda(e) = \begin{cases} \tau & \text{if } e = (X, (e_0, e_1)) \\ \lambda_0(e_0) & \text{if } e = (X, (e_0, *)) \text{ or } e = (e_0, *) \\ \lambda_1(e_1) & \text{if } e = (X, (*, e_1)) \text{ or } e = (*, e_1) \end{cases}$

$\mathsf{Init} = \{(X, e) \mid X \cup \{e\} \subseteq \mathsf{Init}_\times\} \cup (E_{roll} \cap \mathsf{Init}_\times)$

$k(e) = \begin{cases} k_0(e_0) & \text{if } e = (X, (e_0, *)) \\ k_1(e_1) & \text{if } e = (X, (*, e_1)) \\ k_0(e_0) & \text{if } e = (X, (e_0, e_1)) \quad \textit{– note that } k_0(e_0) = k_1(e_1) \end{cases}$

$\{\!|(\nu\ \gamma)P|\!\}_l = \langle (E, F, \mapsto, \rhd, \lambda, \mathsf{Act}), \mathsf{Init}, k\rangle$ where:

$\{\!|P|\!\} = \langle (E, F, \mapsto, \rhd, \lambda_P, \mathsf{Act}_P), \mathsf{Init}, k\rangle \qquad \mathsf{Act} = \mathsf{Act}_P \cup \{\mathsf{roll\ bound}\} \setminus \{\mathsf{roll}\ \gamma\}$

For all $e \in E$, $\lambda(e) = \begin{cases} \lambda_P(e) & \text{if } \lambda_P(e) \neq \mathsf{roll}\ \gamma \\ \mathsf{roll\ bound} & \text{if } \lambda_P(e) = \mathsf{roll}\ \gamma \end{cases}$

$\{\!|P \setminus A|\!\}_l = \left\langle \mathcal{E} \upharpoonright \rho(A \cup \overline{A}), \mathsf{Init} \cap \rho(A \cup \overline{A}), k \upharpoonright \rho(A \cup \overline{A}) \right\rangle$ where $\{\!|P|\!\}_l = \langle \mathcal{E}, \mathsf{Init}, k\rangle$

We next show that process $P$ has a transition $P \xrightarrow{\mu} P'$ if and only if $P$ and $P'$ correspond to isomorphic event structures, and there exists a $\mu$-labelled transition from the initial state of $P$'s event structure to the initial state of $P'$'s event structure.

**Theorem 7.5.** *Let $P$ be a consistent Roll-CCSK process such that $\{\!|P|\!\} = \langle \mathcal{E}, \mathsf{Init}, k\rangle$, $\mathcal{E} = (E, F, \mapsto, \rhd, \lambda, \mathsf{Act})$, $\mathsf{Init}$ is conflict-free, and $C_{er}(\mathcal{E}) = (E, F, \mathsf{C}, \rightarrow)$. Then*

1. *if there exists a process $P'$ with $\{\!|P'|\!\} = \left\langle \mathcal{E}', \mathsf{Init}', k' \right\rangle$ and a transition $P \xrightarrow{\mu_\gamma[m]} P'$ then there exists a transition $\mathsf{Init} \xrightarrow{\{e\}} X$ and an isomorphism $f : \mathcal{E} \to \mathcal{E}'$ such that $\lambda(e) = \mu$, $f \circ k' = k[e \mapsto m]$, and $f(X) = \mathsf{Init}'$;*
2. *and if there exists a transition $\mathsf{Init} \xrightarrow{\{e\}} X$ then there exists a process $P'$ with $\{\!|P'|\!\} = \left\langle \mathcal{E}', \mathsf{Init}', k' \right\rangle$, a transition $P \xrightarrow{\mu_\gamma[m]} P'$, and an isomorphism $f : \mathcal{E} \to \mathcal{E}'$ such that $\lambda(e) = \mu$, $f \circ k' = k[e \mapsto m]$, and $f(X) = \mathsf{Init}'$.*

We then prove the same correspondence for start roll transitions.

**Proposition 7.6.** *Let $P$ be a consistent Roll-CCSK process such that $\{\!|P|\!\} = \langle \mathcal{E}, \mathsf{Init}, k\rangle$, $\mathcal{E} = (E, F, \mapsto, \rhd, \lambda, \mathsf{Act})$, $\mathsf{Init}$ is conflict-free, and $C_{er}(\mathcal{E}) = (E, F, \mathsf{C}, \rightarrow)$. Then*

1. *if there exists a process $P'$ with $\{\!|P'|\!\} = \left\langle \mathcal{E}', \mathsf{Init}', k' \right\rangle$ and a transition $P \overset{\text{start roll } \gamma}{\rightsquigarrow} P'$ then there exists a transition $\mathsf{Init} \xrightarrow{\{e\}} X$ and an isomorphism $f : \mathcal{E} \to \mathcal{E}'$ such that $\lambda(e) = \text{start roll } \gamma$, $f \circ k' = k$, and $f(X) = \mathsf{Init}'$;*
2. *and if there exists a transition $\mathsf{Init} \xrightarrow{\{e\}} X$ then there exists a process $P'$ with $\{\!|P'|\!\} = \left\langle \mathcal{E}', \mathsf{Init}', k' \right\rangle$, a transition $P \overset{\text{start roll } \gamma}{\rightsquigarrow} P'$, and an isomorphism $f : \mathcal{E} \to \mathcal{E}'$ such that $\lambda(e) = \text{start roll } \gamma$, $f \circ k' = k$, and $f(X) = \mathsf{Init}'$.*

We finally show that a process $P$ can make a roll $\gamma$ transition if and only if the REBES corresponding to $P$ can perform a roll $\gamma$ event, followed by reversing all the events corresponding to actions and start roll's with tags causally dependent on $\gamma$, and then finally reversing the roll $\gamma$ event.

**Theorem 7.7.** *Let $P$ be a consistent process with $\{\!|P|\!\} = \langle \mathcal{E}, \mathsf{Init}, k\rangle$, $\mathcal{E} = (E, F, \mapsto, \rhd, \lambda, \mathsf{Act})$, $C_{er}(\mathcal{E}) = (E, F, \mathsf{C}, \rightarrow)$, and $\mathsf{Init}$ conflict-free, and let $\rho \in \{\text{roll } \gamma, \text{bound roll}\}$ be a roll label. Then*

1. *if there exists a process $P'$ with $\{\!|P'|\!\} = \left\langle \mathcal{E}', \mathsf{Init}', k' \right\rangle$ and a transition $P \overset{\rho}{\rightsquigarrow} P'$, then there exist events $e_r$ and $e_0, e_1, \ldots e_n$ such that $\mathsf{Init} \xrightarrow{\{e_r\}} X_0 \xrightarrow{\{\underline{e_0}\}} X_1 \ldots \xrightarrow{\{\underline{e_n}\}} X_{n+1} \xrightarrow{\{\underline{e_r}\}} X_{done}$ and there exists an isomorphism $f : \mathcal{E} \to \mathcal{E}'$ such that $\lambda(e_r) = \rho$, $\{e_0, e_1, \ldots e_n\} = \{e \mid \exists \gamma'.\gamma \leq_P \gamma'$ and either $\lambda(e)_{\gamma'}[k(e)]$ occurs in $P$ or $\lambda(e) = \text{start roll } \gamma'$ and $\text{rolling } \gamma'$ occurs in $P\}$, $f \circ k' = k \upharpoonright \{e \mid f(e) \in \mathsf{Init}'\}$, and $f(X_{done}) = \mathsf{Init}'$;*
2. *and if there exist events $e_r$ and $e_0, e_1, \ldots e_n$ such that $\mathsf{Init} \xrightarrow{\{e_r\}} X_0 \xrightarrow{\{\underline{e_0}\}} X_1 \ldots \xrightarrow{\{\underline{e_n}\}} X_{n+1} \xrightarrow{\{\underline{e_r}\}} X_{done}$ then there exists a process $P'$ with $\{\!|P'|\!\} = \left\langle \mathcal{E}', \mathsf{Init}', k' \right\rangle$ and a transition $P \overset{\rho}{\rightsquigarrow} P'$ and there exists an isomorphism $f : \mathcal{E} \to \mathcal{E}'$ such that $\lambda(e_r) = \rho$, $\{e_0, e_1, \ldots e_n\} = \{e \mid \exists \gamma'.\gamma \leq_P \gamma'$ and either $\lambda(e)_{\gamma'}[k(e)]$ occurs in $P$ or $\lambda(e) = \text{start roll } \gamma'$ and $\text{rolling } \gamma'$ occurs in $P\}$, $f \circ k' = k \upharpoonright \{e \mid f(e) \in \mathsf{Init}'\}$, and $f(X_{done}) = \mathsf{Init}'$.*

## 8 Conclusion

We have defined a category of reversible bundle event structures, and used the causal subcategory to model uncontrolled CCSK. Unlike previous work giving a truly concurrent semantics of a reversible process calculus using rigid families [6] or configuration structures [1], we have used the way CCSK handles past actions to generate both the event structure and the initial state directly from the process, rather than needing to first undo past actions to get the original process and from there the rigid family or configuration structure, and then redo the actions to get the initial state.

We have proposed a variant of CCSK called Roll-CCSK, which uses the rollback described in [9] to control its reversibility. We have defined a category of reversible extended bundle event structures, which use asymmetric rather than symmetric conflict, and used this category to model Roll-CCSK. Unlike in the case of CCSK, when modelling rollbacks in Roll-CCSK we use *non-causal* reversible event structures.

We have proved operational correspondence between the operational and event structure semantics of both CCSK (Theorem 4.10) and Roll-CCSK (Theorems 7.5 and 7.7).

*Future work:* We would like to provide event structure semantics for other reversible calculi. These mostly handle past actions using separate memories, which may prove challenging, particularly if we wish to avoid basing the semantics on finding the fully reversed process.

We also intend to explore the relationship between equivalences of processes and equivalences of event structures.

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# A Section 3

## A.1 BES

**Definition A.1 (Bundle Event Structures [11]).** *A* bundle event structure *(BES) is a triple* $\mathcal{E} = (E, \mapsto, \sharp)$ *where:*

1. *$E$ is the set of events;*
2. *$\mapsto \subseteq 2^E \times E$ is the bundle set, satisfying $X \mapsto e \Rightarrow \forall e_1, e_2 \in X.(e_1 \neq e_2 \Rightarrow e_1 \sharp e_2)$;*
3. *$\sharp \subseteq E \times E$ is the irreflexive; and symmetric conflict relation.*

**Definition A.2 (BES configuration [11]).** *Given a BES $\mathcal{E} = (E, \mapsto, \sharp)$, a configuration of $\mathcal{E}$ is a set $X \subseteq E$ such that:*

1. *$X$ is conflict-free;*
2. *there exists a sequence $e_1, \ldots, e_n$ ($n \geq 0$), such that $X = \{e_1, \ldots, e_n\}$ and for all $i$, $1 \leq i \leq n$, if $Y \mapsto e_{i+1}$ then $\{e_1, \ldots, e_i\} \cap Y \neq \emptyset$.*

A category of BESs has not, to our knowledge, been defined, so we define a BES-morphism in Definition A.3. We want to say that the events of $E_0$ can behave the same way as those they synchronise with in $E_1$, but the bundle sets mean this is a bit trickier to describe than in other event structures. If we said that $f(X) \mapsto f(e)^*$ implies $X \mapsto e^*$, we would be requiring $X' \mapsto e^*$ for every $X' = X \cup X''$ where $e \in X'' \Rightarrow f(e) = \bot$, and by extension $e \sharp e'$ if $f(e) = f(e') = \bot$. As we consider this too restrictive, we came up with the constraint seen in Definition A.3.

**Definition A.3 (BES-morphism).** *Given BESs $\mathcal{E}_0 = (E_0, \mapsto_0, \sharp_0)$ and $\mathcal{E}_1 = (E_1, \mapsto_1, \sharp_1)$, a BES-morphism from $\mathcal{E}_0$ to $\mathcal{E}_1$ is a partial function $f : E_0 \to E_1$ such that and for all $e, e' \in E_0$:*

1. *if $f(e) \sharp_1 f(e')$ then $e \sharp_0 e'$;*
2. *if $f(e) = f(e') \neq \bot$ and $e \neq e'$ then $e \sharp_0 e'$;*
3. *for $X_1 \subseteq E_1$ if $X_1 \mapsto_1 f(e)$ then there exists $X_0 \subseteq E_0$ such that $X_0 \mapsto_0 e$, $f(X_0) \subseteq X_1$, and if $e' \in X_0$ then $f(e') \neq \bot$.*

We show that BES-morphisms preserve configurations.

**Proposition A.4.** *Given BESs $\mathcal{E}_0 = (E_0, \mapsto_0, \sharp_0)$ and $\mathcal{E}_1 = (E_1, \mapsto_1, \sharp_1)$ and BES-morphism $f : E_0 \to E_1$, if $X \subseteq E_0$ is a configuration of $\mathcal{E}_0$, then $f(X)$ is a configuration of $\mathcal{E}_1$*

*Proof.* We show that $f(X)$ fulfils the conditions of Definition A.2:

1. For any $e, e' \in X$, if $f(e) \sharp_1 f(e')$, then $e \sharp_0 e'$, and therefore if $X$ is conflict-free then $f(X)$ is conflict-free.
2. There exists a sequence $e_1, \ldots, e_n$ ($n \geq 0$), such that $X = \{e_1, \ldots, e_n\}$ and for all $i$, $1 \leq i \leq n$, if $Y \mapsto e_{i+1}$ then $\{e_1, \ldots, e_i\} \cap Y \neq \emptyset$. Obviously $f(X_) = \{f(e_1), \ldots, f(e_n)\}$, and for all $i$, if $Y_1 \mapsto f(e_{i+1})$, then there exists $Y_0$ such that $Y_0 \mapsto e_{i+1}$, $f(Y_0) \subseteq Y_1$, and if $e' \in Y_0$, then $f(e') \neq \bot$. Since $Y_0 \cap \{e_1, \ldots, e_i\} \neq \emptyset$, we obviously get that $Y_1 \cap \{f(e_1), \ldots, f(e_i)\} \neq \emptyset$.

**Proposition A.5.** ***BES*** *consisting of BESs and BES-morphisms is a category*

*Proof.* Partial functions are associative and $f(e) = e$ functions as a an identity arrow, so we need to show that the morphisms are composable:

If $\mathcal{E}_0 = (E_0, \mapsto_0, \sharp_0)$, $\mathcal{E}_1 = (E_1, \mapsto_1, \sharp_1)$, and $\mathcal{E}_2 = (E_2, \mapsto_2, \sharp_2)$ are BESs and $f : E_0 \to E_1$ and $g : E_1 \to E_2$ are morphisms, we show that $f \circ g : E_0 \to E_2$ is also a morphism:

1. If $g(f(e)) \sharp_2 g(f(e'))$ then $f(e) \sharp_1 f(e')$, and therefore $e \sharp_0 e'$.
2. If $g(f(e)) = g(f(e'))$ and $e \neq e'$, then either $f(e) = f(e')$, in which case $e \sharp_0 e'$, or $f(e) \neq f(e')$, in which case $f(e) \sharp_1 f(e')$, and therefore $e \sharp_0 e'$.
3. If $X_2 \mapsto_2 g(f(e))$ then there exist $X_1 \subseteq E_1$ and $X_0 \subseteq E_0$ such that $X_1 \mapsto_1 f(e)$, $X_0 \mapsto_0 e$, $g(X_1) \subseteq X_2$, $f(X_0) \subseteq X_1$, if $e_1 \in X_1$ then $g(e_1) \neq \bot$, and if $e_0 \in X_0$ then $f(e_0) \neq \bot$. This means that $g(f(X_0)) \subseteq X_2$, and if $e_0 \in X_0$ then $g(f(e_0)) \neq \bot$.

We also define a product in this category in Definition A.6.

**Definition A.6 (Product of BESs).** *Let $\mathcal{E}_0 = (E_0, \mapsto_0, \sharp_0)$ and $\mathcal{E}_1 = (E_1, \mapsto_1, \sharp_1)$ be bundle event structures. Their product $\mathcal{E}_0 \times \mathcal{E}_1$ is the BES $\mathcal{E} = (E, \mapsto, \sharp)$ defined by:*

1. *$E = E_0 \times_* E_1 = \{(e, *) \mid e \in E_0\} \cup \{(*, e) \mid e \in E_1\} \cup \{(e, e') \mid e \in E_0$ and $e' \in E_1\}$;*
2. *projections $\pi_0, \pi_1$ are defined so that for $(e_0, e_1) \in E$, $\pi_i((e_0, e_1) = e_i$.*
3. *for any $e \in E$, $X \subseteq E$, $X \mapsto e$ iff there exists $i \in \{0, 1\}$ and $X_i \subseteq E_i$ such that $X_i \mapsto \pi_i(e)$ and $X = \{e' \in E \mid \pi_i(e') \in X_i\}$;*
4. *for any $e, e' \in E$, $e \sharp e'$ iff there exists $i \in \{0, 1\}$ such that $\pi_i(e) \sharp_i \pi_i(e')$, or $\pi_i(e) = \pi_i(e') \neq \bot$ and $\pi_{1-i}(e) \neq \pi_{1-i}(e')$.*

**Proposition A.7.** *Given BESs $\mathcal{E}_0 = (E_0, \mapsto_0, \sharp_0)$ and $\mathcal{E}_1 = (E_1, \mapsto_1, \sharp_1)$, $\mathcal{E}_0 \times \mathcal{E}_1 = (E, \mapsto, \sharp)$ is a product.*

*Proof.* We first show that $\pi_0$ and $\pi_1$ are morphisms:

1. If $\pi_i(e) \sharp_i \pi_i(e')$, then obviously $e \sharp e'$.
2. If $\pi_i(e) = \pi_i(e')$ and $e \neq e'$, then $\pi_{1-i}(e) \neq \pi_{1-i}(e')$, and therefore $e \sharp e'$.
3. If $X_i \mapsto \pi_i(e)$, then $\{e' \in E \mid \pi_i(e') \in X_i\} \mapsto e$. Clearly $\pi_i(\{e' \in E \mid \pi_i(e') \in X_i\}) = X_i$, and for all $e' \in \{e' \in E \mid \pi_i(e') \in X_i\})$, $\pi_i(e') \neq \bot$.
4. If $X$ is a configuration of $\mathcal{E}_0 \times \mathcal{E}_1$, then we show that $\pi_i(X)$ satisfies the requirements of a configuration of $\mathcal{E}_i$:
   (a) As shown above $\pi_i(e) \sharp_i \pi_i(e') \Rightarrow e \sharp e'$, meaning $X$ conflict free implies $\pi_i(X)$ conflict-free.
   (b) If there exists a sequence $e_1, \ldots, e_n$ such that $X = \{e_1, \ldots, e_n\}$ and for all $j$, $1 \leq j \leq n$, if $Y \mapsto e_{j+1}$ then $\{e_1, \ldots, e_j\} \cap Y \neq \emptyset$, then $\pi_i(X) = \{\pi_i(e_1), \ldots, \pi_i(e_n)\}$ and if $\pi_i(e_{j+1}) \neq \bot$, then whenever $Y_i \mapsto \pi_i(e_{j+1})$, $\{e' \in E \mid \pi_i(e') \in Y_i\} \mapsto e_{j+1}$, meaning $\{e' \in E \mid \pi_i(e') \in Y_i\} \cap \{e_1, \ldots, e_j\} \neq \emptyset$. Therefore, we must get $Y_i \cap \{\pi_i(e_1), \ldots, \pi_i(e_j)\} \neq \pi_i(\emptyset) = \emptyset$.

We then show that for any BES, $\mathcal{E}_2 = (E_2, \mapsto_2, \sharp_2)$, if there exist morphisms $f_0 : \mathcal{E}_2 \to \mathcal{E}_0$ and $f_1 : \mathcal{E}_2 \to \mathcal{E}_1$, then there exists a unique morphism $f : \mathcal{E}_2 \to \mathcal{E}$, such that $f_2 \circ \pi_0 = f_0$ and $f_2 \circ \pi_1 = f_1$. Since BES-morphisms are functions, they are all unique.

We define $f$ by $f(e) = (f_0(e), f_1(e))$, meaning the morphisms clearly commute as described above, and prove it to be a morphism:

1. If $f(e) \sharp f(e')$ then there exists $i \in \{0, 1\}$ such that either $\pi_i(f(e)) \sharp_i \pi_i(f(e'))$, in which case clearly $f_i(e) \sharp_i f_i(e')_i$, and therefore $e \sharp_2 e'$, or $\pi_i(f(e))) = \pi_i(f(e')) \neq \bot$ and $\pi_{1-i}(f(e)) \neq \pi_{1-i}(f(e'))$, in which case $f_i(e) = f_i(e') \neq \bot$, and $e \neq e'$, meaning $e \sharp_2 e'$.

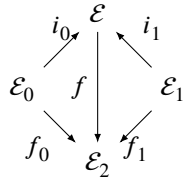
2. If $f(e) = f(e') \neq \bot$ then $f_0(e) = f_0(e') \neq \bot$ or $f_1(e) = f_1(e') \neq \bot$, meaning if $e \neq e'$ then $e \sharp_2 e'$.
3. For $X \subseteq E$, if $X \mapsto f(e)$, then there exists $i \in \{0,1\}$ and $X_i \subseteq E_i$ such that $X_i \mapsto \pi_i(e)$ and $X = \{e' \in E \mid \pi_i(e') \in X_i\}$. And since $X_i \mapsto f_i(e)$, there exists $X_2 \subseteq E_2$ such that $X_2 \mapsto_2 e$, $f_i(X_2) \subseteq X_i$, and if $e' \in X_2$ then $f_i(e') \neq \bot$. Clearly $f(X_2) \subseteq X$.
4. If $X$ is a configuration of $\mathcal{E}_2$, then we show that $f(X)$ satisfies the requirements of a configuration of $\mathcal{E}_0 \times \mathcal{E}_1$:
   (a) As shown above, if $f(e) \sharp f(e')$, then $e \sharp_2 e'$, meaning if $X$ is conflict-free, then $f(X)$ is conflict-free.
   (b) If there exists a sequence $e_1, \ldots, e_n$ such that $X = \{e_1, \ldots, e_n\}$ and for all $j$, $1 \leq j \leq n$, if $Y \mapsto e_{j+1}$ then $\{e_1, \ldots, e_j\} \cap Y \neq \emptyset$, then $f(X) = \{f(e_1), \ldots, f(e_n)\}$ and if $f(e_{j+1}) \neq \bot$, then whenever $Y' \mapsto f(e_{j+1})$, $\{e' \in E \mid f(e') \in Y'\} \mapsto e_{j+1}$, meaning $\{e' \in E \mid f(e') \in Y'\} \cap \{e_1, \ldots, e_j\} \neq \emptyset$. Therefore, we must get $Y' \cap \{f(e_1), \ldots, f(e_j)\} \neq f(\emptyset) = \emptyset$.

**Definition A.8 (BES coproduct).** *Given BESs $\mathcal{E}_0 = (E_0, \mapsto_0, \sharp_0)$ and $\mathcal{E}_1 = (E_1, \mapsto_1, \sharp_1)$, their coproduct $\mathcal{E}_0 + \mathcal{E}_1 = (E, \mapsto, \sharp)$ where:*

- $E = \{(0,e) \mid e \in E_0\} \cup \{(1,e) \mid e \in E_1\}$;
- *injections $i_0$, $i_1$ are defined so that for $e \in E_j$, $i_j(e) = (j,e)$ for $j \in \{0,1\}$;*
- $X \mapsto (j,e)$ *iff for all* $(j',e') \in X$, $j = j'$ *and* $i_j(X) \mapsto_j e$;
- $(j,e) \sharp (j',e')$ *iff* $j \neq j'$ *or* $e \sharp_j e'$.

**Proposition A.9.** *If $\mathcal{E}_0$ and $\mathcal{E}_1$ are BESs, then $\mathcal{E}_0 + \mathcal{E}_1$ is their coproduct.*

*Proof.* Obviously $\mathcal{E}$ is a BES, and $i_0$ and $i_1$ are morphisms, so we simply need to prove that if there exists a BES $\mathcal{E}_2 = (E_2, \mapsto_2, \sharp_2)$ and morphisms $f_0 : \mathcal{E}_0 \to \mathcal{E}_2$ and $f_1 : \mathcal{E}_1 \to \mathcal{E}_2$, then there exists a unique BES-morphism $f : \mathcal{E} \to \mathcal{E}_2$ such that the following commutes:




Since $E_0 + E_1$, $i_0$, and $i_1$ make up a coproduct in the category of sets and partial functions, $f$ must be unique.

We define $f$ as $f((j,e)) = f_j(e)$ and prove it to be a morphism:

- If $f(e)\#_2 f(e')$ then $e = (j,e_j)$, $e' = (j',e_{j'})$ $f_j(e_j) = f(e)$, $f_{j'}(e_{j'}) = f(e')$, and either $j \neq j'$ or $j = j'$. If $j \neq j'$, then obviously $e \sharp e'$. If $j = j'$, then $f_j(e_j) \sharp_2 f_j(e_{j'})$, meaning $e_j \sharp_j e_{j'}$, and therefore $e \sharp e'$.
- If $f(e) = f(e') \neq \bot$ then $e = (j,e_j)$ and $e' = (j',e_{j'})$ and $f_j(e_j) = f(e) = f(e') = f_{j'}(e_{j'})$.
  If $j = j'$ then $e \neq e'$ means that $e_j \neq e_{j'}$, which means that $e_j \#_j e_{j'}$ and therefore $e\#e'$.
  If $j \neq j'$, then by definition $e\#e'$.

– If $X_2 \mapsto f(e)$, then $e = (j, e_j)$, and there exists $X_j$ such that $X_j \mapsto_j e_j$, $f_j(X_j) \subseteq X_2$, and if $e'_j \in X_j$ then $f_j(e'_j) \neq \bot$. This means $\{(j, e'_j) \mid e'_j \in X_j\} \mapsto e$, $f(\{(j, e'_j) \mid e'_j \in X_j\}) \subseteq X_2$, and if $e' \in \{(j, e'_j) \mid e'_j \in X_j\}$ then $e' \neq \bot$.

The diagram obviously commutes.

### A.2 Product and Coproduct

**Proposition A.10.** *Given RBESs $\mathcal{E}_0 = (E_0, F_0, \mapsto_0, \sharp_0, \rhd_0)$ and $\mathcal{E}_1 = (E_1, F_1, \mapsto_1, \sharp_1, \rhd_1)$, $\mathcal{E}_0 \times \mathcal{E}_1$ is a product.*

*Proof.* Similar to proof of Proposition A.7.

**Proposition A.11.** *If $\mathcal{E}_0$ and $\mathcal{E}_1$ are RBESs, then $\mathcal{E}_0 + \mathcal{E}_1$ is their coproduct.*

*Proof.* Similar to proof of Proposition A.9.

### A.3 Proposition A.12

**Proposition A.12.** ***RBES*** *consisting of RBESs and RBES-morphisms is a category.*

*Proof.* Partial functions are associative, and $f(e) = e$ functions as a an identity arrow, and the morphisms are obviously composable.

### A.4 Proof of Proposition 3.8

*Proof.* 1. There exists a trace $\emptyset \xrightarrow{\{e_0^*\}} C_0 \xrightarrow{\{e_1^*\}} C_1 \ldots \xrightarrow{\{e_n^*\}} C_n$ where $C_n = C$. Clearly $e_0^* = e_0$, and $C_0$ is forwards-reachable, and we will show that if $C_j$ is forwards-reachable for $0 \leq j \leq i$, then $C_{i+1}$ is forwards-reachable.
If $e_{i+1}^* = e_{i+1}$, then obviously $C_{i+1}$ is forwards-reachable.
If $e_{i+1}^* = \underline{e_i}$, then $C_{i+1} = C_{i-1}$, which is obviously forwards-reachable.
If $e_{i+1}^* = \underline{e_j}$ for some $0 \leq j < i$, then for all $0 \leq j' \leq i$, since $\not\rhd \underline{e_j}$, there does not exist $X \subseteq E$ such that $e_j \in X$ and $X \mapsto e_{j'}$. This means obviously $\emptyset \xrightarrow{\{e_0^*\}} C_0 \xrightarrow{\{e_1^*\}} C_1 \ldots \xrightarrow{\{e_{j-1}^*\}} C_{j-1} \xrightarrow{\{e_{j+1}^*\}} C_{j+1} \setminus \{e_j\} \ldots \xrightarrow{\{e_i^*\}} C_{i+1}$.

2. For any forwards-reachable configuration $X \in \mathsf{C}$ and $A, B \subseteq F$, if $X \xrightarrow{A \cup \underline{B}} (X \cup A) \setminus B$ then $(X \cup A) \setminus B \xrightarrow{B \cup \underline{A}} X$ according to Definition 3.5:
   (a) obvious
   (b) For all $e \in A$ and $e' \in E$, if $e' \rhd \underline{e}$, then either $e' \sharp e$, or there exists $X' \subseteq E$ such that $X' \mapsto e'$ and $e \in X'$.
   If $e' \sharp e$, then, as $X \cup A$ is conflict-free, $e' \notin X \cup A$.
   If there exists $X' \subseteq E$ such that $X' \mapsto e'$ and $e \in X'$ then for all $e'' in X' \setminus \{e\}$ we know $e'' \sharp e$, meaning $e'' \notin X \cup A$. This means $X \cap X' = \emptyset$, and therefore for all $X'' \subseteq X$, $X'' \not\xrightarrow{e'}$, and consequently $e' \notin X \cup A$.

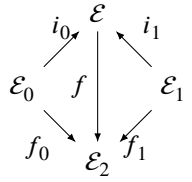
(c) For all $e \in B$ and $X' \subseteq E$, if $X' \mapsto e$, then, since $X$ is forwards-reachable, $X' \cap X \neq \emptyset$. If $X' \cap X \setminus B = \emptyset$, then there exists $e' \in X' \cap B$. But this means $e \rhd \underline{e'}$, conflicting with $X \xrightarrow{A \cup \underline{B}}$.
(d) For all $e \in A$ and $X' \subseteq E$, if $X' \mapsto \underline{e}$, then $e \in X'$.

# B Section 4

## B.1 Proposition B.1

**Proposition B.1.** *If $\mathcal{E}_0$ and $\mathcal{E}_1$ are LRBESs, then $\mathcal{E}_0 \& \mathcal{E}_1 = \mathcal{E}$ with injections $i_0$ and $i_1$ such that $i_j(j, e) = e$ is their coproduct.*

*Proof.* Obviously $\mathcal{E}$ is an LRBES, and $i_0$ and $i_1$ are morphisms, so we simply need to prove that if there exists an LRBES $\mathcal{E}_2 = (E_2, F_2, \mapsto_2, \sharp_2, \rhd_2, \lambda_2, \mathsf{Act}_2)$ and morphisms $f_0 : \mathcal{E}_0 \to \mathcal{E}_2$ and $f_1 : \mathcal{E}_1 \to \mathcal{E}_2$, then there exists a unique LRBES-morphism $f : \mathcal{E} \to \mathcal{E}_2$ such that the following commutes:




Since $E_0 + E_1$, $i_0$, and $i_1$ make up a coproduct in the category of sets and partial functions, $f$ must be unique.

We define $f$ as $f((j, e)) = f_j(e)$ and prove it to be a morphism. Since $(E, F, \mapsto, \sharp, \rhd) = (E_0, F_0, \mapsto_0, \sharp_0, \rhd_0) + (E_1, F_1, \mapsto_1, \sharp_1, \rhd_1)$, we know $f : (E, F, \mapsto, \sharp, \rhd) \to (E_2, F_2, \mapsto_2, \sharp_2, \rhd_2)$ is an RBES-morphism, and by definition $\lambda((e, j)) = \lambda_j(e) = \lambda_2(f_j(e))$.

The diagram obviously commutes.

## B.2 $\leq$ is a Complete Partial Order

$\leq$ is clearly a partial order with the empty LRBES as its minimum.

**Proposition B.2.** *Any $\omega$-chain $\mathcal{E}_0 \leq \mathcal{E}_1 \leq \mathcal{E}_2 \ldots$ has a least upper bound $\mathcal{E} = (E, F, \mapsto, \sharp, \rhd, \lambda, \mathsf{Act})$ where:*

1. $E = \bigcup_{n \in \omega} E_n$;
2. $F = \bigcup_{n \in \omega} F_n$;
3. $X \mapsto e^*$ *if for all $n \in \omega$ such that $e \in E_n$, $(X \cap E_n) \mapsto e^*$;*
4. $\sharp = \bigcup_{n \in \omega} \sharp_n$;
5. $\rhd = \bigcup_{n \in \omega} \rhd_n$;
6. $\lambda(e) = l$ *if there exists $n \in \omega$ such that $\lambda_n(e) = l$;*
7. $\mathsf{Act} = \bigcup_{n \in \omega} \mathsf{Act}_n$;

This means that, given a set of events $A$, with $\mathbb{E}_A$ being the set of LRBESs $(E, F, \mapsto, \sharp, \rhd, \lambda, \mathsf{Act})$ such that $E \subseteq A$, $(\mathbb{E}_A, \leq)$ is a complete partial order. We then need to show that our operations are monotonic.

*Proof.* Clearly $\mathcal{E}$ is an LRBES, and for all $i \in \omega$, $\mathcal{E}_i \leq \mathcal{E}$:

We therefore know that $\mathcal{E}$ is an upper bound of the chain. We now show that $\mathcal{E}$ is the least upper bound of the chain: Given an upper bound $\mathcal{E}'$, it is obvious that $E \subseteq E'$, $F = E \cap F'$, and if $X \mapsto e^*$ then, for all $n \in \omega$, if $e \in E_n$ then $X \cap E_n \mapsto_n e^*$, meaning since $\mathcal{E}_n \leq \mathcal{E}'$ there exists $X'_n \subseteq E'$ such that $X'_n \mapsto e^*$ and $X'_n \cap E_n = X_n$, and since for all $n' \geq n$, $\mathcal{E}_{n'} \leq \mathcal{E}'$ and $e \in E_{n'}$, $X'_n \cap E_{n'} \mapsto_{n'} e^*$. This means clearly $X'_n \cap E = \bigcup_{n \in \omega} X_n = X$. And for $e \in E$, if $X' \mapsto' e'^*$, then for all $n \in \omega$ such that $e \in E_n$ $X' \cap E_n \mapsto_n e^*$, meaning $X' \cap E \mapsto e^*$. Similar arguments apply to $\sharp$, $\rhd$, $\lambda$, and $\mathsf{Act}$.

This means, clearly $\mathcal{E} \leq \mathcal{E}'$, and $\mathcal{E}$ is the least upper bound of the chain.

### B.3 Proof of Proposition 4.7

*Proof.* It should be obvious that all the operations used for defining the $\mathcal{E}$ and $\mathcal{E}'$ are monotonic, so clearly $\mathcal{E}' \leq \mathcal{E}$, and since $P$ has been generated from a standard process, we cannot have any $\alpha[m]$ inside a recursion, as it would have to have been unfolded first.

### B.4 Proof of Proposition 4.8

*Proof.* We prove this by induction in $P$:

- Suppose $P = 0$. Then $\mathcal{E}$ is empty, and therefore obviously causal.
- Suppose $P = P_0 + P_1$, $e \in E$ and $e' \in F$. Then if $e \rhd \underline{e'}$, then there exists $i \in \{0, 1\}$ such that either $e \rhd_i e'$ or $e \in E_i$ and $e' \in F_{1-i}$. By induction, $e \rhd_i e'$ means there exists an $X_i \subseteq E_i$ such that $X_i \mapsto_i e$ and $e' \in X_i$. As $X_i \mapsto_i e$, we get $X_i \mapsto e$. And $E_i \times E_{1-i} \subseteq \sharp$.
  If there exists an $X \subseteq E$ such that $X \mapsto e$ and $e' \in X$, then there exists an $i \in \{0, 1\}$ such that $X \mapsto_i e$. That by induction we get $e \rhd_i \underline{e'}$, implying $e \rhd \underline{e'}$.
  $X \mapsto \underline{e'}$ if and only if there exists an $i \in \{0, 1\}$ such that $X \mapsto_i \underline{e'}$. By induction, this means $e' \in X$.
- Suppose $P = \alpha.P'$, $e \in E$ and $e' \in F$. Then if $e \rhd \underline{e'}$, then either $e \rhd' \underline{e'}$, or $e' = e_\alpha$ and $e \in E'$. If $e \rhd' \underline{e'}$, then by induction there exists an $X \subseteq E'$ such that $X \mapsto' e$ and $e' \in X$, and $X \mapsto e$. If $e' = e_\alpha$ and $e \in E'$ then we know $\{e_\alpha\} \mapsto e$.
  If there exists an $X \subseteq E$ such that $X \mapsto e$ and $e' \in X$, then either $X \mapsto' e$, or $X = \{e_\alpha\}$ and $e \in E'$. If $X \mapsto' e$, then by induction we get $e \rhd' \underline{e'}$, and therefore $e \rhd \underline{e'}$. If $X = \{e_\alpha\}$ and $e \in E'$, then we know $e \rhd \underline{e_\alpha}$.
  $X \mapsto \underline{e'}$ if and only if $X \mapsto' \underline{e'}$ or $e' = e_\alpha$ and $X = \{e_\alpha\}$. By induction, if $X \mapsto' \underline{e'}$ then $e' \in X$.
- Suppose $P = \alpha[m].P'$. Then the proof is similar to the previous case.
- Suppose $P = P_0 \mid P_1$, $e \in E$ and $e' \in F$. Then if $e \rhd \underline{e'}$, then there exists an $i \in \{0, 1\}$, such that $\pi_i(e) \rhd_i \pi_i(e')$. By induction, this means there exists an $X_i \subseteq E_i$ such that $X_i \mapsto_i \pi_i(e)$ and $\pi_i(e') \in X_i$. This means $\{e'' \in E \mid \pi_i(e'') \in X_i\} \mapsto e$, and obviously $e' \in \{e'' \in E \mid \pi_i(e'') \in X_i\}$.

If there exists an $X \subseteq E$ such that $X \mapsto e$ and $e' \in X$, then there exists an $i \in \{0,1\}$ and $X_i \subseteq E_i$ such that $X_i \mapsto_i \pi_i(e)$ and $X = \{e'' \in E \mid \pi_i(e'') \in X_i\}$, meaning $\pi_i(e') \in X_i$. By induction we get $\pi_i(e) \rhd_i \underline{\pi_i(e')}$, and therefore $e \rhd \underline{e'}$.

$X \mapsto \underline{e'}$ if and only if there exists $i \in \{0,1\}$ and $X_i \subseteq E_i$ such that $X_i \mapsto_i \pi_i(e')$ and $X = \{e'' \in E \mid \pi_i(e'') \in X_i\}$. By induction, since $X_i \mapsto_i \pi_i(e')$ we know $\pi_i(e') \in X_i$, meaning clearly $e' \in X$.

- Suppose $P = P' \setminus A$, $e \in E$ and $e' \in F$. Then $X \mapsto e'^*$ if and only if $X \mapsto' e'^*$ and $e \rhd \underline{e'}$ if and only if $e \rhd' \underline{e'}$. The proof is trivial induction.
- Suppose $P = P'[f]$. Then the proof is trivial induction.

### B.5 Proof of Proposition 4.9

*Proof.* We say that $\mathcal{E} = (E, F, \mapsto, \sharp, \rhd, \lambda, \mathsf{Act})$ and $\mathcal{E}' = (E', F', \mapsto', \sharp', \rhd', \lambda', \mathsf{Act}')$ and do a case analysis on the Structural congruence rules:

$P = X \mid Y$ **and** $P' = Y \mid X$**:** Products are unique up to isomorphism and

$$f(e) = \begin{cases} (e_Y, e_X) & \text{if } e = (e_X, e_Y) \\ (e_Y, *) & \text{if } e = (*, e_Y) \\ (*, e_X) & \text{if } e = (e_X, *) \end{cases}$$ clearly fulfils the conditions other conditions.

$P = X \mid (Y \mid Z)$ **and** $P' = (X \mid Y) \mid Z$**:** Products are associative up to isomorphism, and $f((e_X, (e_Y, e_Z)) = ((e_X, e_Y), e_Z)$ clearly fulfills the other conditions.

$P = P' \mid 0$**:** If $f((e, *)) = e$, then this clearly holds.

$P = X + Y$ **and** $P' = Y + X$**:** Coproducts are unique up to isomorphism, and $f(e) = e$ clearly fulfil the other conditions.

$P = (X + Y) + Z$ **and** $P' = (X + Y) + Z$**:** Coproducts are associative up to isomorphism, and $f(e) = e$ clearly fulfils the other conditions.

$P = P' + 0$**:** Clearly $\mathcal{E}_P = (E' \cup \emptyset, F' \cup \emptyset, \mapsto' \cup \emptyset, \sharp' \cup \emptyset, \rhd' \cup \emptyset, \lambda', \mathsf{Act}' \cup \emptyset)$, $\mathsf{Init} = \mathsf{Init}'$, and $k = k'$.

$P = A\left\langle \tilde{b} \right\rangle$ **and** $P' = P_A\{{}^{\tilde{b}}/_{\tilde{a}}\}$ **where** $A\langle \tilde{a} \rangle = P_A$**:** Obvious

### B.6 Lemma B.3

**Lemma B.3 (standard).** *Given a reachable process $P$ such that $\llbracket P \rrbracket = \langle \mathcal{E}, \mathsf{Init}, k \rangle$, we have* $\mathsf{std}(P)$ *if and only if* $\mathsf{Init} = \emptyset$.

*Proof.* As the only rule which can add events to an empty $\mathsf{Init}$ is $\{\!|\alpha[m].P|\!\}$, clearly $\mathsf{Init} = \emptyset$ if $\mathsf{std}(P)$.

If $I = \emptyset$, then clearly we cannot have any $\alpha[m]$ in $P$, which are not guarded by a restriction on $\alpha$. But if such a restricted communication has occurred in $P$, then there must exist a parallel $\overline{a}[m]$ inside the same restriction, meaning the corresponding event $(e_\alpha, e_{\overline{\alpha}})$ has the label $\tau$, not $\alpha$, and would therefore be in $\mathsf{Init}$. Therefore we must have $\mathsf{std}(P)$.

### B.7 Lemma B.4

**Lemma B.4 (Reachable).** *Given a reachable process $P$, and $\{\!|P|\!\} = \langle \mathcal{E}, \mathsf{Init}, k \rangle$, then* $\mathsf{Init}$ *is conflict-free in* $\mathcal{E}$.

*Proof.* We show this by structural induction in $P$, which we can do because of Proposition 5.5 of [19], the proof of which is not affected by adding definitions.

- Suppose $P = 0$. Then $\mathsf{Init} = \emptyset$.
- Suppose $P = \alpha.P'$. Then $\mathsf{Init} = \mathsf{Init}'$, and therefore $\mathsf{Init}$ is conflict-free.
- Suppose $P = \alpha[m].P'$. Then $\mathsf{Init} = \mathsf{Init}' \cup \{e_\alpha\}$, $\mathsf{Init}'$ is conflict-free, and therefore $\mathsf{Init}$ is clearly conflict-free.
- Suppose $P = P_1 + P_2$. Then $\mathsf{Init} = \mathsf{Init}_1 \cup \mathsf{Init}_2$ and, since $P$ is reachable from a standard process, either $\mathsf{Init}_1 = \emptyset$ or $\mathsf{Init}_2 = \emptyset$, and both $\mathsf{Init}_1$ and $\mathsf{Init}_2$ are conflict-free. Therefore, $\mathsf{Init}$ is conflict-free.
- Suppose $P = P_1 \mid P_2$. Then, since $P$ is reachable from a standard process, each key appears at most once in $P_1$ and once in $P_2$. Additionally, $\mathsf{Init}_1$ is conflict-free and $\mathsf{Init}_2$ is conflict-free, meaning $\mathsf{Init} = \{(e_0, *) \mid e_0 \in \mathsf{Init}_0 \text{ and } \nexists e_1 \in \mathsf{Init}_1.\lambda_0(e_0) = \overline{\lambda_1(e_1)} \text{ and } k_0(e_0) = k_1(e_1)\} \cup \{(*, e_1) \mid e_1 \in \mathsf{Init}_1 \text{ and } \nexists e_0 \in \mathsf{Init}_0.\lambda_0(e_0) = \overline{\lambda_1(e_1)} \text{ and } k_0(e_0) = k_1(e_1)\} \cup \{(e_0, e_1) \mid e_0 \in \mathsf{Init}_0, e_1 \in \mathsf{Init}_1, k_0(e_0) = k_1(e_1)\}$ is conflict-free.
- Suppose $P = P' \setminus A$. Then $\mathsf{Init}'$ is conflict-free, and $\mathsf{Init} \subseteq \mathsf{Init}'$, meaning $\mathsf{Init}$ is conflict-free.
- Suppose $P = P'[f]$. then $\mathsf{Init} = \mathsf{Init}'$, which is conflict-free.

### B.8 Proof of Theorem 4.10

*Proof.* We first prove that if there exists a $P'$ and a transition $P \xrightarrow{\mu[m]} P'$ then there exists a transition $\mathsf{Init} \xrightarrow{\{e\}} X$ and an isomorphism $f : \mathcal{E} \to \mathcal{E}'$ such that $\lambda(e) = \mu$, $f \circ k' = k[e \mapsto m]$, and $f(X) = \mathsf{Init}'$. We prove this by inductions on the transition rules:

- Suppose $P = \alpha.Q$, $P' = \alpha[m].Q$, $\mu = \alpha$, and $\mathsf{std}(Q)$. Then there exist. $\mathcal{E}_Q$ and $e_\alpha$ such that:
  $e_\alpha \notin E_Q$,
  $\{\!|Q|\!\} = \langle \mathcal{E}_Q, \mathsf{Init}, k \rangle$,
  $E = E_Q \cup \{e_\alpha\}$,
  $F = F_Q \cup \{e_\alpha\}$,
  $X \mapsto e^*$ if $X \mapsto_Q e^*$ or $X = \{e_\alpha\}$ and $e^* = e \in E_Q$,
  $\sharp = \sharp_Q$,
  $\rhd = \rhd_Q \cup (E_Q \times \{\underline{e_\alpha}\})$,
  $\mathsf{Act} = \mathsf{Act}_Q \cup \{\alpha\}$,
  for all $e \in E$, $\lambda(e) = \begin{cases} \lambda_Q(e) & \text{if } e \in E_Q \\ \alpha & \text{if } e = e_\alpha \end{cases}$,
  and if $\mathsf{Init} \neq \emptyset$ then for all $e \in E$, $\{e\} \mapsto e$ and for all $e \in F$, $e \rhd \underline{e}$.

There also exists $\mathcal{E}'_Q$ and $e'_\alpha$ such that: $e'_\alpha \notin E'_Q$
$\{\!|Q|\!\} = \left\langle \mathcal{E}'_Q, \mathsf{Init}'_Q, k'_Q \right\rangle$,
$E' = E'_Q \cup \{e'_\alpha\}$,
$F' = F'_Q \cup \{e'_\alpha\}$,
$X \mapsto' e^*$ if $X \mapsto'_Q e^*$ or $X = \{e'_\alpha\}$ and $e^* = e \in E'_Q$,
$\sharp' = \sharp'_Q$,
$\rhd' = \rhd'_Q \cup (E'_Q \times \{\underline{e'_\alpha}\})$,
$\mathsf{Act}' = \mathsf{Act}'_Q \cup \{\alpha\}$,
for all $e \in E'$, $\lambda'(e) = \begin{cases} \lambda'_Q(e) & \text{if } e \in E'_Q \\ \alpha & \text{if } e = e'_\alpha \end{cases}$,
$\mathsf{Init}' = \mathsf{Init}'_Q \cup \{e'_\alpha\}$,
and $k'(e) = \begin{cases} k'_Q(e) & \text{If } e \in \mathsf{Init}'_Q \\ m & \text{If } e = e'_\alpha \end{cases}$.
As $\mathcal{E}_Q$ and $\mathcal{E}'_Q$ have been generated by the same process, we have an isomorphism $f_Q : \mathcal{E}_Q \to \mathcal{E}'_Q$. We say that $f = f_Q[e_\alpha \mapsto e'_\alpha]$, which is obviously an isomorphism. Since Init is conflict-free and $\sharp = \sharp_Q$, $X = \mathsf{Init} \cup \{e_\alpha\}$ is conflict free, and therefore a configuration of $C_{br}(\mathcal{E})$. And since no $X' \subseteq E$ exists such that $X' \mapsto e_\alpha$, we get $\mathsf{Init} \xrightarrow{\{e_\alpha\}} \{e_\alpha\}$, and clearly $\lambda(e_\alpha) = \alpha$ and $k(f(e)) = m$.

– Suppose that $P = \alpha[n].Q$, $P' = \alpha[n].Q'$, $Q \xrightarrow{\mu[m]} Q'$, and $m \neq n$. Then there exist $\mathcal{E}_Q$ and $e_\alpha$ such that:
$e_\alpha \notin E_Q$,
$\{\!|Q|\!\} = \left\langle \mathcal{E}_Q, \mathsf{Init}_Q, k_Q \right\rangle$,
$E = E_Q \cup \{e_\alpha\}$,
$F = F_Q \cup \{e_\alpha\}$,
$X \mapsto e^*$ if $X \mapsto_Q e^*$ or $X = \{e_\alpha\}$ and $e^* = e \in E_Q$,
$\sharp = \sharp_Q$,
$\rhd = \rhd_Q \cup (E_Q \times \{\underline{e_\alpha}\})$,
$\mathsf{Act} = \mathsf{Act}_Q \cup \{\alpha\}$,
for all $e \in E$, $\lambda(e) = \begin{cases} \lambda_Q(e) & \text{if } e \in E_Q \\ \alpha & \text{if } e = e_\alpha \end{cases}$,
$\mathsf{Init} = \mathsf{Init}_Q \cup \{e_\alpha\}$,
and $k(e) = \begin{cases} k_Q(e) & \text{If } e \in \mathsf{Init}_Q \\ m & \text{If } e = e_\alpha \end{cases}$.
And there exist $\mathcal{E}_{Q'}$ and $e'_\alpha$ such that:
$e'_\alpha \notin E_{Q'}$,
$\{\!|Q'|\!\} = \left\langle \mathcal{E}_{Q'}, \mathsf{Init}_{Q'}, k_{Q'} \right\rangle$,
$E' = E_{Q'} \cup \{e'_\alpha\}$,
$F' = F_{Q'} \cup \{e'_\alpha\}$,
$X \mapsto' e^*$ if $X \mapsto_{Q'} e^*$ or $X = \{e'_\alpha\}$ and $e^* = e \in E_{Q'}$,
$\sharp' = \sharp_{Q'}$,

$\rhd' = \rhd_{Q'} \cup (E_{Q'} \times \{\underline{e'_\alpha}\})$,
$\mathsf{Act}' = \mathsf{Act}_{Q'} \cup \{\alpha\}$,
for all $e \in E'$, $\lambda'(e) = \begin{cases} \lambda_{Q'}(e) & \text{if } e \in E_{Q'} \\ \alpha & \text{if } e = e'_\alpha \end{cases}$,
$\mathsf{Init}' = \mathsf{Init}_{Q'} \cup \{e'_\alpha\}$,
and $k'(e) = \begin{cases} k_{Q'}(e) & \text{If } e \in \mathsf{Init}_{Q'} \\ m & \text{If } e = e'_\alpha \end{cases}$.

By induction, we get an isomorphism $f_Q : \mathcal{E}_Q \to \mathcal{E}_{Q'}$ and a transition $\mathsf{Init}_Q \xrightarrow{\{e\}} X_Q$ in $C_{br}(\mathcal{E}_Q)$ such that $\lambda_Q(e) = \mu$, $k_{Q'}(f_Q(e)) = m$, and $f_Q(X_Q) = \mathsf{Init}_{Q'}$.
We define $f = f_Q[e_\alpha \mapsto e'_\alpha]$. Since $\mathsf{Init}_Q$ and $X_Q$ are conflict-free in $\mathcal{E}_Q$, $\mathsf{Init}_Q \cup \{e_\alpha\} = \mathsf{Init}$ and $X_Q \cup \{e_\alpha\} = X$ are configurations of $C_{rb}(\mathcal{E})$, and clearly $\mathsf{Init} \xrightarrow{\{e\}} X$.

– Suppose $P = Q \mid R$, $P' = Q' \mid R$, $Q \xrightarrow{\mu[m]} Q'$, and $\mathsf{fsh}[m](R)$. Then there exist $\mathcal{E}_Q$ and $\mathcal{E}_R$ such that
$\{\!|Q|\!\} = \left\langle \mathcal{E}_Q, \mathsf{Init}_Q, k_Q \right\rangle$,
$\{\!|R|\!\} = \langle \mathcal{E}_R, \mathsf{Init}_R, k_R \rangle$,
$\mathcal{E}_P = \mathcal{E}_Q || \mathcal{E}_R$,
$\mathsf{Init} = \{(e_Q, *) \mid e_Q \in \mathsf{Init}_Q \text{ and } \nexists e_R \in \mathsf{Init}_R . \lambda_Q(e_Q) = \overline{\lambda_R(e_R)} \text{ and } k_Q(e_Q) = k_R(e_R)\} \cup \{(*, e_R) \mid e_R \in \mathsf{Init}_R \text{ and } \nexists e_Q \in \mathsf{Init}_Q . \lambda_Q(e_Q) = \overline{\lambda_R(e_R)} \text{ and } k_Q(e_Q) = k_R(e_R)\} \cup \{(e_Q, e_R) \mid e_Q \in \mathsf{Init}_Q, e_R \in \mathsf{Init}_R, k_Q(e_Q) = k_R(e_R)\}$,
and $k(e) = \begin{cases} k_Q(e_Q) & \text{If } e = (e_Q, *) \\ k_R(e_R) & \text{If } e = (*, e_R) \\ k_Q(e_Q) & \text{If } e = (e_Q, e_R) \end{cases}$.
We also have $\left\langle \mathcal{E}', \mathsf{Init}', k' \right\rangle$ similarly made up of some $\left\langle \mathcal{E}_{Q'}, \mathsf{Init}_{Q'}, k_{Q'} \right\rangle$ and $\left\langle \mathcal{E}'_R, \mathsf{Init}'_R, k'_R \right\rangle$ such that $\{\!|Q'|\!\} = \left\langle \mathcal{E}_{Q'}, \mathsf{Init}_{Q'}, k_{Q'} \right\rangle$ and $\{\!|R|\!\} = \left\langle \mathcal{E}'_R, \mathsf{Init}'_R, k'_R \right\rangle$.
We clearly have isomorphisms $f_Q : \mathcal{E}_Q \to \mathcal{E}_{Q'}$ and $f_R : \mathcal{E}_R \to \mathcal{E}'_R$ and a transition $\mathsf{Init}_Q \xrightarrow{\{e_Q\}} X_Q$ of $C_{rb}(\mathcal{E}_Q)$ such that $\lambda_Q(e) = \mu$, $k_{Q'}(f_Q(e_Q)) = m$, and $f_Q(X_Q) = \mathsf{Init}_{Q'}$.
Since $\mathsf{Init}$ is conflict-free and $X_Q$ is conflict-free in $\mathcal{E}_Q$, clearly $\mathsf{Init} \cup \{(e_Q, *)\} = X$ is conflict-free, and $\mathsf{Init} \xrightarrow{(e_Q, *)} X$.
We define our isomorphism as $f(e) = \begin{cases} (f_Q(e'), *) & \text{if } e = (e', *) \\ (*, f_R(e')) & \text{if } e = (*, e') \\ (f_Q(e'), f_R(e'')) & \text{if } e = (e', e'') \end{cases}$. And, since $\mathsf{fsh}[m](R)$, $f(X) = \mathsf{Init}'$. And the rest of the proof is straightforward.

– Suppose $P = Q \mid R$, $P' = Q' \mid R'$, $Q \xrightarrow{\alpha[m]} Q'$, $R \xrightarrow{\overline{\alpha}[m]} R'$, and $\mu = \tau$. Then there exist $\mathcal{E}_Q$ and $\mathcal{E}_R$ such that
$\{\!|Q|\!\} = \left\langle \mathcal{E}_Q, \mathsf{Init}_Q, k_Q \right\rangle$,
$\{\!|R|\!\} = \langle \mathcal{E}_R, \mathsf{Init}_R, k_R \rangle$,
$\mathcal{E}_P = \mathcal{E}_Q || \mathcal{E}_R$,

$\mathsf{Init} = \{(e_Q, *) \mid e_Q \in \mathsf{Init}_Q \text{ and } \nexists e_R \in \mathsf{Init}_R . \lambda_Q(e_Q) = \overline{\lambda_R(e_R)} \text{ and } k_Q(e_Q) = k_R(e_R)\} \cup \{(*, e_R) \mid e_R \in \mathsf{Init}_R \text{ and } \nexists e_Q \in \mathsf{Init}_Q . \lambda_Q(e_Q) = \overline{\lambda_R(e_R)} \text{ and } k_Q(e_Q) = k_R(e_R)\} \cup \{(e_Q, e_R) \mid e_Q \in \mathsf{Init}_Q, e_R \in \mathsf{Init}_R, k_Q(e_Q) = k_R(e_R)\}$,
and $k(e) = \begin{cases} k_Q(e_Q) & \text{If } e = (e_Q, *) \\ k_R(e_R) & \text{If } e = (*, e_R) \\ k_Q(e_Q) & \text{If } e = (e_Q, e_R) \end{cases}$.
We also have $\left\langle \mathcal{E}', \mathsf{Init}', k' \right\rangle$ similarly made up of some $\left\langle \mathcal{E}_{Q'}, \mathsf{Init}_{Q'}, k_{Q'} \right\rangle$ and $\left\langle \mathcal{E}_{R'}, \mathsf{Init}_{R'}, k_{R'} \right\rangle$ such that $\{\!|Q'|\!\} = \left\langle \mathcal{E}_{Q'}, \mathsf{Init}_{Q'}, k_{Q'} \right\rangle$ and $\{\!|R'|\!\} = \langle \mathcal{E}_{R'}, \mathsf{Init}_{R'}, k_{R'} \rangle$.
By induction, we have isomorphisms $f_Q : \mathcal{E}_Q \to \mathcal{E}_{Q'}$ and $f_R : \mathcal{E}_R \to \mathcal{E}_{R'}$ and transitions $\mathsf{Init}_Q \xrightarrow{\{e_Q\}} X_Q$ of $C_{rb}(\mathcal{E}_Q)$ such that $\lambda_Q(e) = \alpha$, $k_{Q'}(f_Q(e_Q)) = m$, and $f_Q(X_Q) = \mathsf{Init}_{Q'}$, and $\mathsf{Init}_R \xrightarrow{\{e_R\}} X_R$ of $C_{rb}(\mathcal{E}_R)$ such that $\lambda_R(e) = \overline{\alpha}$, $k_{R'}(f_R(e_R)) = m$, and $f_R(X_R) = \mathsf{Init}_{R'}$.
We define our isomorphism as $f(e) = \begin{cases} (f_Q(e'), *) & \text{if } e = (e', *) \\ (*, f_R(e')) & \text{if } e = (*, e') \\ (f_Q(e'), f_R(e'')) & \text{if } e = (e', e'') \end{cases}$.
We know that $\mathsf{Init}$, $X_Q$, and $X_R$ are conflict-free, so the only way $\mathsf{Init} \cup \{(e_Q, e_R)\}$ has conflict is if $\mathsf{Init}_Q$ or $\mathsf{Init}_R$ contains an event with the key $m$, which we know from Lemma 5.2 of [19], whcih is not affected by definitions, as they cannot define processes with past actions, is not possible. The rest of the proof is straightforward.

– Suppose $P = Q + R$, $P' = Q' + R$, $Q \xrightarrow{\mu[m]} Q'$, and $\mathsf{std}(R)$. Then there exist $\mathcal{E}_Q$ and $\mathcal{E}_R$ such that:
$\{\!|Q|\!\} = \left\langle \mathcal{E}_Q, \mathsf{Init}_Q, k_Q \right\rangle$,
$\{\!|R|\!\} = \langle \mathcal{E}_R, \mathsf{Init}_R, k_R \rangle$,
$E = E_Q \cup E_R$,
$F = F_Q \cup F_R$,
$X \mapsto e^*$ if there exists $i \in \{Q, R\}$ such that $X \mapsto_i e^*$,
$\sharp = \sharp_Q \cup \sharp_R \cup (E_Q \times E_R) \cup (E_R \times E_Q)$,
$\rhd = \rhd_Q \cup \rhd_R \cup (E_Q \times \underline{F_R}) \cup (E_R \times \underline{F_Q})$,
$\mathsf{Act} = \mathsf{Act}_Q \cup \mathsf{Act}_R$,
for all $e \in E$, $i \in \{Q, R\}$, $\lambda(e) = \lambda_i(e)$ if $e \in E_i$,
$\mathsf{Init} = \mathsf{Init}_Q \cup \mathsf{Init}_R$,
for $i \in \{Q, R\}$, $k(e) = k_i(e)$ if $e \in \mathsf{Init}_i$,
If $\mathsf{Init}_Q \neq \emptyset$ and $\mathsf{Init}_Q \neq \emptyset$ then for all $e \in E$, $\{e\} \mapsto e$ and for all $e \in F$, $e \rhd \underline{e}$.
We also have $\left\langle \mathcal{E}', \mathsf{Init}', k' \right\rangle$ similarly made up of some $\left\langle \mathcal{E}_{Q'}, \mathsf{Init}_{Q'}, k_{Q'} \right\rangle$ and $\left\langle \mathcal{E}'_R, \mathsf{Init}'_R, k'_R \right\rangle$ such that $\{\!|Q'|\!\} = \left\langle \mathcal{E}_{Q'}, \mathsf{Init}_{Q'}, k_{Q'} \right\rangle$ and $\{\!|R|\!\} = \left\langle \mathcal{E}'_R, \mathsf{Init}'_R, k'_R \right\rangle$.
We clearly have isomorphisms $f_Q : \mathcal{E}_Q \to \mathcal{E}_{Q'}$ and $f_R : \mathcal{E}_R \to \mathcal{E}'_R$ and a transition $\mathsf{Init}_Q \xrightarrow{\{e\}} X$ of $C_{rb}(\mathcal{E}_Q)$ such that $\lambda_Q(e) = \mu$, $k_{Q'}(f_Q(e)) = m$, and $f(X) = \mathsf{Init}_{Q'}$.
We define our isomorphism $f(e) = \begin{cases} f_Q(e) & \text{if } e \in E_Q \\ f_R(e) & \text{if } e \in E_R \end{cases}$.
Since $\mathsf{std}(R)$, $\mathsf{Init}_R = \emptyset$, and therefore $\mathsf{Init} = \mathsf{Init}_Q$, which is conflict-free end therefore a configuration. Obviously $\mathsf{Init} \xrightarrow{\{e\}} X$ in $C_{rb}(\mathcal{E})$, and the rest follows.

- Suppose $P = Q \setminus A$, $P' = Q' \setminus A$, $Q \xrightarrow{\mu[m]} Q'$, and $\mu \notin A \cup \overline{A}$. Then there exists $\mathcal{E}_Q$ such that:
  $\{\!|Q|\!\} = \langle \mathcal{E}_Q, \mathsf{Init}_Q, k_Q \rangle$
  $\mathcal{E} = \mathcal{E}_Q \upharpoonright \{e \mid \lambda_Q(e) \notin A \cup \overline{A}\}$
  $\mathsf{Init} = \mathsf{Init}_Q \cap \{e \mid \lambda_Q(e) \notin A \cup \overline{A}\}$
  $k = k_Q \upharpoonright \{e \mid \lambda_Q(e) \notin A \cup \overline{A}\}$
  And there exists $\mathcal{E}_{Q'}$ such that:
  $\{\!|Q'|\!\} = \langle \mathcal{E}_{Q'}, \mathsf{Init}_{Q'}, k_{Q'} \rangle$
  $\mathcal{E}' = \mathcal{E}_{Q'} \upharpoonright \{e \mid \lambda_{Q'}(e) \notin A \cup \overline{A}\}$
  $\mathsf{Init}' = \mathsf{Init}_{Q'} \cap \{e \mid \lambda_{Q'}(e) \notin A \cup \overline{A}\}$
  $k' = k_{Q'} \upharpoonright \{e \mid \lambda_{Q'}(e) \notin A \cup \overline{A}\}$

  By inductions we have an isomorphism $f_Q : \mathcal{E}_Q \to \mathcal{E}_{Q'}$ and a transition $\mathsf{Init}_Q \xrightarrow{\{e_Q\}} X_Q$ or $C_{rb}(\mathcal{E}_Q)$ such that $\lambda_Q(e) = \mu$, $k_{Q'}(f_Q(e_Q)) = m$, and $f_Q(X_Q) = \mathsf{Init}_{Q'}$. We define our isomorphism as $f_Q \upharpoonright \{e \mid \lambda_{Q'}(e) \notin A \cup \overline{A}\}$. And since $\lambda(e) \notin A \cup \overline{A}$, the rest of the proof is straightforward.
- Suppose $P = Q[f']$, $P' = Q'[f']$, $Q \xrightarrow{\nu[m]} Q'$, and $f'(\nu) = \mu$. Then there exist $\lambda_Q$ and $\mathsf{Act}_Q$ such that
  $\{\!|Q|\!\} = \langle (E, F, \mapsto, \sharp, \rhd, \lambda_Q, \mathsf{Act}_Q), \mathsf{Init}, k \rangle$,
  $\mathsf{Act} = f'(\mathsf{Act}_Q)$
  and $\lambda = f' \circ \lambda_Q$.
  And there exist $\lambda_{Q'}$ and $\mathsf{Act}_{Q'}$ such that
  $\{\!|Q'|\!\} = \langle (E', F', \mapsto', \sharp', \rhd', \lambda_{Q'}, \mathsf{Act}_{Q'}), \mathsf{Init}', k' \rangle$,
  $\mathsf{Act}' = f'(\mathsf{Act}_{Q'})$
  and $\lambda = f' \circ \lambda_{Q'}$.

  By induction, we get an isomorphism $f_Q : \mathcal{E}_Q \to \mathcal{E}_{Q'}$ and a transition $\mathsf{Init} \xrightarrow{\{e\}} X$ in $C_{rb}(\mathcal{E}_Q)$ such that $\lambda_Q(e) = \nu$, $k'(f'(e)) = m$, and $f(X) = \mathsf{Init}'$. We define our isomorphisms $f = f_Q$, and the rest of the proof is straightforward.
- Suppose $P \equiv Q$, $P' \equiv Q'$, and $Q \xrightarrow{\mu[m]} Q'$. Then the result follows from induction and Proposition 4.9.

We then prove that if there exists a transition $\mathsf{Init} \xrightarrow{\{e\}} X$ then there exist a $P'$ and a transition $P \xrightarrow{\mu[m]} P'$ and an isomorphism $f : \mathcal{E} \to \mathcal{E}'$ and such that $\lambda(e) = \mu$, $f \circ k' = k[e \mapsto m]$, and $f(X) = \mathsf{Init}'$.

- Suppose $P = 0$. Then $E = \emptyset$, and obviously no transitions exist in $C_{br}(\mathcal{E})$.
- Suppose $P = \alpha.Q$. Then $\{e_\alpha\} \mapsto e'$ for all $e' \in E \setminus \{e_\alpha\}$, meaning by definition $e = e_\alpha$. In addition, since $P$ is reachable, clearly $\mathsf{std}(P)$ meaning $\mathsf{Init} = \emptyset$. This means we get $P \xrightarrow{\alpha[m]} \alpha[m].Q$ for some fresh $m$, and the isomorphisms are similar to this case in the first part of the proof.
- Suppose $P = \alpha[n].Q$ and $\{\!|Q|\!\} = \langle \mathcal{E}_Q, \mathsf{Init}_Q, k_Q \rangle$. Then $e_\alpha \in \mathsf{Init}$, and clearly $\mathsf{Init}_Q \xrightarrow{e}_Q X_Q$, meaning there exists a key $m$ and a transition $Q \xrightarrow{\lambda(e)[m]} Q'$, such

that $\{\!|Q'|\!\} = \left\langle \mathcal{E}_{Q'}, \mathsf{Init}_{Q'}, k_{Q'} \right\rangle$ and there exists an isomorphism $f_Q : \mathcal{E}_Q \to \mathcal{E}_{Q'}$ such that $k_{Q'}(f_Q(e)) = m$ and $f_Q(X_Q) = \mathsf{Init}_{Q'}$. If $m \neq n$, then $P \xrightarrow{\lambda(e)[m]} \alpha[m].Q'$. Otherwise, we can chose a fresh $m$ and still get a transition. We define our isomorphism as $f = f_Q[e_\alpha \mapsto e'_\alpha]$ and the rest of the proof is straightforward.

– Suppose $P = P_0 + P_1$, $\{\!|P_0|\!\} = \langle \mathcal{E}_0, \mathsf{Init}_0, k_0 \rangle$, $C_{br}(\mathcal{E}_0) = (E_0, F_0, \mathsf{C}_0, \to_0)$, $\{\!|P_1|\!\} = \langle \mathcal{E}_1, \mathsf{Init}_1, k_1 \rangle$, and $C_{br}(\mathcal{E}_1) = (E_1, F_1, \mathsf{C}_1, \to_1)$. Then either $\mathsf{Init}_0 \xrightarrow{e}_0 X_0$ and $\mathsf{Init}_1 = \emptyset$, or $\mathsf{Init}_1 \xrightarrow{e}_1 X_1$ and $\mathsf{Init}_0 = \emptyset$.

  If $\mathsf{Init}_0 \xrightarrow{e}_0 X_0$, then there exists a key $m$ and a transition $P_0 \xrightarrow{\lambda_0(e)[m]} P'_0$, such that $\left\{\!\left|P'_0\right|\!\right\} = \left\langle \mathcal{E}'_0, \mathsf{Init}'_0, k'_0 \right\rangle$ and there exists an isomorphism $f_0 : \mathcal{E}_0 \to \mathcal{E}'_0$ such that $k'_0(f_0(e)) = m$ and $f_0(X_0) = \mathsf{Init}'_0$. Then, since $\mathsf{Init}_1 = \emptyset$ means $\mathsf{std}(P_1)$, $P \xrightarrow{\lambda(e)[m]} P'_0 + P_1$, and the isomorphisms are similar to this case in the first part of the proof.

  If $\mathsf{Init}_1 \xrightarrow{e}_1 X_1$, then the proof is similar.

– Suppose $P = P_0 \mid P_1$, $\{\!|P_0|\!\} = \langle \mathcal{E}_0, \mathsf{Init}_0, k_0 \rangle$, $C_{br}(\mathcal{E}_0) = (E_0, F_0, \mathsf{C}_0, \to_0)$, $\{\!|P_1|\!\} = \langle \mathcal{E}_1, \mathsf{Init}_1, k_1 \rangle$, and $C_{br}(\mathcal{E}_1) = (E_1, F_1, \mathsf{C}_1, \to_1)$. Then either $e = (e_0, *)$, $e = (*, e_1)$, or $e = (e_0, e_1)$.

  If $e = (e_0, *)$, then whenever $X'_0 \mapsto_0 e_0$, we get $\{e' \in E \mid \pi_0(e') \in X'_0\} \mapsto e$. And whenever $\pi_0(e') \sharp_0 \pi_0(e)$, we get $e' \sharp e$. This means $\mathsf{Init}_0$ is conflict-free, $\pi_0(X)$ is conflict-free, and $\mathsf{Init}_0 \xrightarrow{e_0}_0 \pi_0(X)$. There therefore exists a key $m$ and a transition $P_0 \xrightarrow{\lambda_0(e_0)[m]} P'_0$, such that $\left\{\!\left|P'_0\right|\!\right\} = \left\langle \mathcal{E}'_0, \mathsf{Init}'_0, k'_0 \right\rangle$ and there exists an isomorphism $f_0 : \mathcal{E}_0 \to \mathcal{E}'_0$ such that $k'_0(f_0(e_0)) = m$ and $f_0(\pi_0(X)) = \mathsf{Init}'_0$.

  We chose an $m$, which is fresh for $P_1$, and we get $P \xrightarrow{\lambda_0(e_0)[m]} P'_0 \mid P_1$. We define our isomorphism

  $$f(e') = \begin{cases} (f_0(e'_0), *) & \text{if } e' = (e'_0, *) \\ (e' & \text{if } e' = (*, e'_1) \\ (f_0(e'_0), e'_1) & \text{if } e' = (e'_0, e'_1) \end{cases}$$

  . Since $\mathcal{E}' = \mathcal{E}'_0 \times \mathcal{E}_1$, $f$ is an isomorphism, and the rest of the case is straightforward.

  If $e = (e_0, *)$, the argument is similar.

  If $e = (e_0, e_1)$, then for $i \in \{0, 1\}$, whenever $X'_i \mapsto_i e_i$, we get $\{e' \in E \mid \pi_i(e') \in X'_i\} \mapsto e$. And whenever $\pi_i(e') \sharp_i \pi_i(e)$, we get $e' \sharp e$. This means $\mathsf{Init}_i$ is conflict-free, $\pi_i(X)$ is conflict-free, and $\mathsf{Init}_i \xrightarrow{e_0}_i \pi_i(X)$. There therefore exists a key $m_i$ and a transition $P_i \xrightarrow{\lambda_i(e_i)[m_i]} P'_i$, such that $\left\{\!\left|P'_i\right|\!\right\} = \left\langle \mathcal{E}'_i, \mathsf{Init}'_i, k'_i \right\rangle$ and there exists an isomorphism $f_i : \mathcal{E}_i \to \mathcal{E}'_i$ such that $k'_i(f_i(e_i)) = m_i$ and $f_i(\pi_i(X)) = \mathsf{Init}'_i$.

  We say that $m_0 = m_1$ is a fresh $m$, and then since $\lambda_0(e_0) = \overline{\lambda_1(e_1)}$ and $\lambda(e) = \tau$, we get $P \xrightarrow{\lambda(e)[m]} P'_0 \mid P'_1$. We define our isomorphism

  $$f(e') = \begin{cases} (f_0(e'_0), *) & \text{if } e' = (e'_0, *) \\ ((*, f_1(e'_1)) & \text{if } e' = (*, e'_1) \\ (f_0(e'_0), f_1(e'_1)) & \text{if } e' = (e'_0, e'_1) \end{cases}$$

  . Since $\mathcal{E}' = \mathcal{E}'_0 \times \mathcal{E}'_1$, $f$ is an isomorphism, and the rest of the case is straightforward.

- Suppose $P = Q \setminus A$, $\{\!|Q|\!\} = \left\langle \mathcal{E}_Q, \mathsf{Init}, k \right\rangle$, and $C_{br}(\mathcal{E}_Q) = (E_Q, F_Q, \mathsf{C}_Q, \rightarrow_Q)$. Then $\lambda(e) \notin A \cup \overline{A}$ and $\mathsf{Init} \xrightarrow{e}_Q X$, meaning there exists a key $n$ and a transition $Q \xrightarrow{\lambda_Q(e)} Q'$ such that $\{\!|Q'|\!\} = \left\langle \mathcal{E}_{Q'}, \mathsf{Init}_{Q'}, k_{Q'} \right\rangle$, and there exists an isomorphism $f_Q : \mathcal{E}_Q \to \mathcal{E}_{Q'}$ such that $f_Q \circ k_{Q'} = [e \mapsto n]$ and $f_Q(X) = \mathsf{Init}_{Q'}$.
  This means $P \xrightarrow{\lambda_Q(e)} Q' \setminus A$ and the morphisms $f \upharpoonright E$ and $g \upharpoonright \{e' \in E'_Q \mid \lambda'_Q(e') \notin A \cup \overline{A}\}$ clearly fulfil the remaining conditions.
- Suppose $P = Q[f]$, $\{\!|Q|\!\} = \left\langle \mathcal{E}_Q, \mathsf{Init}, k \right\rangle$, and $C_{br}(\mathcal{E}_Q) = (E_Q, F_Q, \mathsf{C}_Q, \rightarrow_Q)$. Clearly $\mathsf{Init} \xrightarrow{e}_Q X$, and $f(\lambda_Q(e)) = \lambda(e)$, and the proof is straightforward.

### B.9 Reverse operational correspondence

**Theorem B.5.** *Let $P$ and $P'$ be processes, $\mu$ be an action, and $m$ be a key such that $\{\!|P|\!\} = \langle \mathcal{E}, \mathsf{Init}, k \rangle$, $\mathcal{E} = (E, F, \mapsto, \sharp, \rhd, \lambda, \mathsf{Act})$, $\mathsf{Init}$ is conflict-free, $C_{br}(\mathcal{E}) = (E, F, \mathsf{C}, \rightarrow)$, and $\{\!|P'|\!\} = \left\langle \mathcal{E}', \mathsf{Init}', k' \right\rangle$. Then there exists a transition $P \xrightsquigarrow{\mu[m]} P'$ if and only if there exists an isomorphism $f : \mathcal{E} \to \mathcal{E}'$ and a transition $\mathsf{Init} \xrightarrow{\{e\}} X$ such that $\lambda(e) = \mu$, $(f \circ k')[e \mapsto m] = k$, and $f(X) = \mathsf{Init}'$.*

*Proof.* Implied by Proposition 4.8, Theorem 4.10, and Corollary 4.11.

## C Section 5

### C.1 EBES

**Definition C.1 (Extended Bundle Event Structure [11]).** *An EBES is a triple $\mathcal{E} = (E, \mapsto, \rhd)$ where:*

1. *$E$ is the set of events;*
2. *$\mapsto \subseteq 2^E \times E$ is the bundle set, satisfying $X \mapsto e \Rightarrow \forall e_1, e_2 \in X.(e_1 \neq e_2 \Rightarrow e_1 \rhd e_2)$;*
3. *$\rhd \subseteq E \times E$ is the asymmetric conflict relation, which is irreflexive.*

**Definition C.2 (EBES configuration [11]).** *Given an EBES $\mathcal{E} = (E, \mapsto, \rhd)$, a configuration of $\mathcal{E}$ is a set $X \subseteq E$ such that there exists a sequence $e_0, \ldots, e_n$ such that:*

1. *$\{e_0, \ldots, e_n\} = X$;*
2. *if $e_i \rhd e_j$ then $j < i$;*
3. *if $X \mapsto e_i$ then $X \cap \{e_0, \ldots, e_{i-1}\} \neq \emptyset$.*

A category of EBESs has not, to our knowledge, been defined, and so we define one with product and coproduct.

**Definition C.3 (EBES-morphism).** *Given EBESs $\mathcal{E}_0 = (E_0, \mapsto_0, \rhd_0)$ and $\mathcal{E}_1 = (E_1, \mapsto_1, \rhd_1)$, a EBES-morphism from $\mathcal{E}_0$ to $\mathcal{E}_1$ is a partial function $f : E_0 \to E_1$ such that and for all $e, e' \in E_0$:*

1. *if* $f(e) \rhd_1 f(e')$ *then* $e \rhd_0 e'$.
2. *if* $f(e) = f(e') \neq \bot$ *and* $e \neq e'$ *then* $e \rhd_0 e'$*;*
3. *for* $X_1 \subseteq E_1$ *if* $X_1 \mapsto_1 f(e)$ *then there exists* $X_0 \subseteq E_0$ *such that* $X_0 \mapsto_0 e$*,* $f(X_0) \subseteq X_1$*, and if* $e' \in X_0$ *then* $f(e') \neq \bot$*;*
4. *for any* $X_0 \subseteq E_0$*, if* $X_0$ *is a configuration of* $\mathcal{E}_0$*, then* $f(X_0)$ *is a configuration of* $\mathcal{E}_1$.

**Proposition C.4.** *Given EBESs* $\mathcal{E}_0 = (E_0, \mapsto_0, \rhd_0)$ *and* $\mathcal{E}_1 = (E_1, \mapsto_1, \rhd_1)$ *and EBES-morphism* $f : E_0 \to E_1$*, if* $X \subseteq E_0$ *is a configuration of* $\mathcal{E}_0$*, then* $f(X)$ *is a configuration of* $\mathcal{E}_1$.

*Proof.* Similar to proof of Proposition A.4.

**Proposition C.5.** ***EBES*** *consisting of EBESs and EBES-morphisms is a category.*

*Proof.* Partial functions are associative and $f(e) = e$ works as a an identity arrow, so we need to show that the morphisms are composable:

If $\mathcal{E}_0 = (E_0, \mapsto_0, \rhd_0)$, $\mathcal{E}_1 = (E_1, \mapsto_1, \rhd_1)$, and $\mathcal{E}_2 = (E_2, \mapsto_2, \rhd_2)$ are EBESs and $f : E_0 \to E_1$ and $g : E_1 \to E_2$ are morphisms, we show that $f \circ g : E_0 \to E_2$ is also a morphism:

1. If $g(f(e)) \rhd_2 g(f(e'))$ then $f(e) \rhd_1 f(e')$, and therefore $e \rhd_0 e'$.
2. If $g(f(e)) = g(f(e'))$ and $e \neq e'$, then either $f(e) = f(e')$, in which case $e \rhd_0 e'$, or $f(e) \neq f(e')$, in which case $f(e) \rhd_1 f(e')$, and therefore $e \rhd_0 e'$.
3. If $X_2 \mapsto_2 g(f(e))$ then there exist $X_1 \subseteq E_1$ and $X_0 \subseteq E_0$ such that $X_1 \mapsto_1 f(e)$, $X_0 \mapsto_0 e$, $g(X_1) \subseteq X_2$, $f(X_0) \subseteq X_1$ and if $e_1 \in X_1$ then $g(e_1) \neq \bot$ and if $e_0 \in X_0$ then $f(e_0) \neq \bot$. This means that $g(f(X_0)) \subseteq X_2$, and if $e_0 \in X_0$ then $g(f(e_0)) \neq \bot$.
4. If $X_0$ is a configuration of $\mathcal{E}_0$ then $f(X_0)$ is a configuration of $\mathcal{E}_1$, and therefore $g(f(X_0))$ is a configurations of $\mathcal{E}_2$.

We also define a product in this category in Definition C.6.

**Definition C.6 (Product of EBESs).** *Let* $\mathcal{E}_0 = (E_0, \mapsto_0, \rhd_0)$ *and* $\mathcal{E}_1 = (E_1, \mapsto_1, \rhd_1)$ *be bundle event structures. Their product* $\mathcal{E}_0 \times \mathcal{E}_1$ *is the EBES* $\mathcal{E} = (E, \mapsto, \rhd)$ *define by:*

1. $E = E_0 \times_* E_1 = \{(e, *) \mid e \in E_0\} \cup \{(*, e) \mid e \in E_1\} \cup \{(e, e') \mid e \in E_0$ *and* $e' \in E_1\}$.
2. *projections* $\pi_0, \pi_1$ *are defined so that for* $(e_0, e_1) \in E$*,* $\pi_i((e_0, e_1) = e_i$.
3. *for any* $e \in E$*,* $X \subseteq E$*,* $X \mapsto e$ *iff there exists* $i \in \{0, 1\}$ *and* $X_i \subseteq E_i$ *such that* $X_i \mapsto \pi_i(e)$ *and* $X = \{e' \in E \mid \pi_i(e') \in X_i\}$.
4. *for any* $e, e' \in E$*,* $e \rhd e'$ *iff there exists* $i \in \{0, 1\}$ *such that* $\pi_i(e) \rhd_i \pi_i(e')$*, or* $\pi_i(e) = \pi_i(e') \neq \bot$ *and* $\pi_{1-i}(e) \neq \pi_{1-i}(e')$.

**Proposition C.7.** *Given EBESs* $\mathcal{E}_0 = (E_0, \mapsto_0, \rhd_0)$ *and* $\mathcal{E}_1 = (E_1, \mapsto_1, \rhd_1)$*,* $\mathcal{E}_0 \times \mathcal{E}_1 = (E, \mapsto, \rhd)$ *is a product.*

*Proof.* We first show that $\pi_0$ and $\pi_1$ are morphisms:

1. If $\pi_i(e) \rhd_i \pi_i(e')$, then obviously $e \rhd e'$.
2. If $\pi_i(e) = \pi_i(e')$ and $e \neq e'$, then $\pi_{1-i}(e) \neq \pi_{1-i}(e')$, and therefore $e \rhd e'$.

3. If $X_i \mapsto \pi_i(e)$, then $\{e' \in E \mid \pi_i(e') \in X_i\} \mapsto e$. Clearly $\pi_i(\{e' \in E \mid \pi_i(e') \in X_i\}) = X_i$, and for all $e' \in \{e' \in E \mid \pi_i(e') \in X_i\})$, $\pi_i(e') \neq \bot$.
4. If $X$ is a configuration of $\mathcal{E}_0 \times \mathcal{E}_1$, then we show that $\pi_i(X)$ satisfies the requirements of a configuration of $\mathcal{E}_i$. We show that if the requirements of Definition C.2 hold for $e_0, \dots e_n$, then they hold for $\pi_i(e_0), \dots, \pi_i(e_n)$:
   (a) Obviously $\{\pi_i(e_0), \dots, \pi_i(e_n)\} = \pi_i(X)$.
   (b) If $\pi_i(e_j) \rhd_i \pi_i(e_{j'})$, then as shown above, $e_j \rhd e_{j'}$, meaning $j' < j$.
   (c) Whenever $Y_i \mapsto \pi_i(e_{j+1})$, we know $\{e' \in E \mid \pi_i(e') \in Y_i\} \mapsto e_{j+1}$, meaning $\{e' \in E \mid \pi_i(e') \in Y_i\} \cap \{e_1, \dots, e_j\} \neq \emptyset$. Therefore, we must get $Y_i \cap \{\pi_i(e_1), \dots, \pi_i(e_j)\} \neq \pi_i(\emptyset) = \emptyset$.

We then show that for any EBES, $\mathcal{E}_2 = (E_2, \mapsto_2, \rhd_2)$, if there exist morphisms $f_0 : \mathcal{E}_2 \to \mathcal{E}_0$ and $f_1 : \mathcal{E}_2 \to \mathcal{E}_1$, then there exists a unique morphism $f : \mathcal{E}_2 \to \mathcal{E}$, such that $f_2 \circ \pi_0 = f_0$ and $f_2 \circ \pi_1 = f_1$. Since EBES-morphisms are functions, they are all unique.

We define $f$ by $f(e) = (f_0(e), f_1(e))$, meaning the morphisms clearly commute as described above, and prove it to be a morphism:

1. If $f(e) \rhd f(e')$ then there exists $i \in \{0, 1\}$ such that either $\pi_i(f(e)) \rhd_i \pi_i(f(e'))$, in which case clearly $f_i(e) \rhd_i f_i(e')_i$, and therefore $e \rhd_2 e'$, or $\pi_i(f(e))) = \pi_i(f(e')) \neq \bot$ and $\pi_{1-i}(f(e)) \neq \pi_{1-i}(f(e'))$, in which case $f_i(e) = f_i(e') \neq \bot$, and $e \neq e'$, meaning $e \rhd_2 e'$.
2. If $f(e) = f(e') \neq \bot$ then $f_0(e) = f_0(e') \neq \bot$ or $f_1(e) = f_1(e') \neq \bot$, meaning if $e \neq e'$ then $e \rhd_2 e'$.
3. For $X \subseteq E$, if $X \mapsto f(e)$, then there exists $i \in \{0, 1\}$ and $X_i \subseteq E_i$ such that $X_i \mapsto \pi_i(e)$ and $X = \{e' \in E \mid \pi_i(e') \in X_i\}$. And since $X_i \mapsto f_i(e)$, there exists $X_2 \subseteq E_2$ such that $X_2 \mapsto_2 e$, $f_i(X_2) \subseteq X_i$, and if $e' \in X_2$ then $f_i(e') \neq \bot$. Clearly $f(X_2) \subseteq X$.
4. If $X$ is a configuration of $\mathcal{E}_2$, then we show that $f(X)$ satisfies the requirements of a configuration of $\mathcal{E}_0 \times \mathcal{E}_1$. We show that if the requirements of Definition C.2 hold for $e_0, \dots e_n$, then they hold for $f(e_0), \dots, f(e_n)$:
   (a) Obviously $\{f(e_0), \dots, f(e_n)\} = f(X)$.
   (b) If $f(e_j) \rhd f(e_{j'})$, then as shown above, $e_j \rhd_2 e_{j'}$, meaning $j' < j$.
   (c) Whenever $Y \mapsto f(e_{j+1})$, we know $\{e' \in E \mid f(e') \in Y\} \mapsto e_{j+1}$, meaning $\{e' \in E \mid f(e') \in Y\} \cap \{e_1, \dots, e_j\} \neq \emptyset$. Therefore, we must get $Y \cap \{f(e_1), \dots, f(e_j)\} \neq f(\emptyset) = \emptyset$.

**Definition C.8 (EBES coproduct).** *Given EBESs $\mathcal{E}_0 = (E_0, \mapsto_0, \rhd_0)$ and $\mathcal{E}_1 = (E_1, \mapsto_1, \rhd_1)$, their coproduct $\mathcal{E}_0 + \mathcal{E}_1 = (E, \mapsto, \rhd)$ where:*

- $E = \{(0, e) \mid e \in E_0\} \cup \{(1, e) \mid e \in E_1\}$;
- *injections $i_0$, $i_1$ are defined so that for $e \in E_j$, $i_j(e) = (j, e)$ for $j \in \{0, 1\}$;*
- $X \mapsto (j, e)$ *iff for all* $(j', e') \in X$, $j = j'$ *and* $i_j(X) \mapsto_j e$;
- $(j, e) \rhd (j', e')$ *iff* $j \neq j'$ *or* $e \rhd_j e'$.

**Proposition C.9.** *If $\mathcal{E}_0$ and $\mathcal{E}_1$ are EBESs, then $\mathcal{E}_0 \times \mathcal{E}_1$ is their coproduct.*

*Proof.* Similar to that of BES coproduct.

### C.2 REBES Category

**Definition C.10 (REBES-morphism).** *Given REBESs $\mathcal{E}_0 = (E_0, F_0, \mapsto_0, \rhd_0)$ and $\mathcal{E}_1 = (E_1, F_1, \mapsto_1, \rhd_1)$, an REBES-morphism from $\mathcal{E}_0$ to $\mathcal{E}_1$ is a partial function $f : E_0 \to E_1$ such that $f(F_0) \subseteq F_1$ and for all $e, e' \in E_0$:*

1. *if $f(e) = f(e')$ and $e \neq e'$ then $e \sharp_0 e'$;*
2. *for $X_1 \subseteq E_1$ if $X_1 \mapsto_1 f(e)^*$ then there exists $X_0 \subseteq E_0$ such that $f(X_0) \subseteq X_1$, if $e' \in X_0$ then $f(e') \neq \bot$, and $X \mapsto_0 e^*$;*
3. *if $f(e) \rhd_1 f(e')^*$ then $e \rhd_0 \underline{e'}^*$.*

**Definition C.11 (Product of REBESs).** *Let $\mathcal{E}_0 = (E_0, F_0, \mapsto_0, \rhd_0)$ and $\mathcal{E}_1 = (E_1, F_1, \mapsto_1, \rhd_1)$ be reversible bundle event structures. Their product $\mathcal{E}_0 \times \mathcal{E}_1$ is the REBES $\mathcal{E} = (E, F, \mapsto, \rhd)$ define by:*

1. *$E = E_0 \times_* E_1 = \{(e, *) \mid e \in E_0\} \cup \{(*, e) \mid e \in E_1\} \cup \{(e, e') \mid e \in E_0$ and $e' \in E_1\}$;*
2. *$F = F_0 \times_* F_1 = \{(e, *) \mid e \in F_0\} \cup \{(*, e) \mid e \in F_1\} \cup \{(e, e') \mid e \in F_0$ and $e' \in F_1\}$;*
3. *projections $\pi_0, \pi_1$ are defined so that for $(e_0, e_1) \in E$, $\pi_i((e_0, e_1) = e_i$;*
4. *for any $e^* \in E \cup \underline{F}$, $X \subseteq E$, $X \mapsto e^*$ iff there exists $i \in \{0, 1\}$ and $X_i \subseteq E_i$ such that $X_i \mapsto \pi_i(e)^*$ and $X = \{e' \in E \mid \pi_i(e') \in X_i\}$;*
5. *for any $e \in E$, $e'^* \in E \cup \underline{F}$, $e \rhd e'^*$ iff there exists $i \in \{0, 1\}$ such that $\pi_i(e) \rhd_i \pi_i(e')^*$.*

**Proposition C.12.** *Given REBESs $\mathcal{E}_0 = (E_0, F_0, \mapsto_0, \rhd_0)$ and $\mathcal{E}_1 = (E_1, F_1, \mapsto_1, \rhd_1)$, $\mathcal{E}_0 \times \mathcal{E}_1 = (E, F, \mapsto, \rhd)$ is a product.*

*Proof.* Similar to proof of Proposition C.7.

**Definition C.13 (REBES coproduct).** *Given REBESs $\mathcal{E}_0 = (E_0, F_0, \mapsto_0, \rhd_0)$ and $\mathcal{E}_1 = (E_1, F_1, \mapsto_1, \rhd_1)$, their coproduct $\mathcal{E}_0 + \mathcal{E}_1 = (E, F, \mapsto, \rhd)$ where:*

- *$E = \{(0, e) \mid e \in E_0\} \cup \{(1, e) \mid e \in E_1\}$;*
- *$F = \{(0, e) \mid e \in F_0\} \cup \{(1, e) \mid e \in F_1\}$;*
- *injections $i_0, i_1$ are defined so that for $e \in E_j$, $i_j(e) = (j, e)$ for $j \in \{0, 1\}$;*
- *$X \mapsto (j, e)^*$ iff for all $(j', e') \in X$, $j = j'$ and $i_j(X) \mapsto_j e^*$;*
- *$(j, e)^* \rhd (j', e')$ iff $j \neq j'$ or $e^* \rhd_j e'$.*

**Proposition C.14.** *If $\mathcal{E}_0$ and $\mathcal{E}_1$ are REBESs, then $\mathcal{E}_0 + \mathcal{E}_1$ is their coproduct.*

**Definition C.15 (From REBES to CS).** *The functor $C_{er}$ : **REBES** $\to$ **CS** is defined as:*

1. *$C_{er}((E, F, \mapsto, \rhd)) = (E, F, \subset, \to)$ where:*
   *(a) $X \in \subset$ if $\rhd$ is well-founded on $X$;*
   *(b) For $X, Y \in \subset$, $A \subseteq E$, and $B \subseteq F$, there exists a transition $X \xrightarrow{A \cup \underline{B}} Y$ if:*
      *i. $Y = (X \setminus B) \cup A$; $X \cap A = \emptyset$; and $B \subseteq X$;*
      *ii. for all $e^* \in A \cup \underline{B}$, if $e' \rhd e^*$ then $e' \notin X \cup A$;*
      *iii. for all $e \in A$ and $X' \subseteq E$, if $X' \mapsto e$ then $X' \cap (X \setminus B) \neq \emptyset$;*
      *iv. for all $e \in B$ and $X' \subseteq E$, if $X' \mapsto \underline{e}$ then $X' \cap (X \setminus (B \setminus \{e\})) \neq \emptyset$.*

2. $C_{er}(f) = f$.

The definition of a causal REBES (Definition C.16) is of course practically identical to that of a CRBES.

**Definition C.16 (Causal REBES).** $(E, F, \mapsto, \sharp, \rhd, \lambda, \mathsf{Act})$ *is a causal REBES (CREBES) if (1) if* $e \rhd \underline{e'}$ *then either* $e \sharp e'$ *or there exists an* $X \subseteq E$ *such that* $X \mapsto e$ *and* $e' \in X$*, (2) if* $e \rhd e'$ *or* $X \mapsto e$ *and* $e' \in X \cap F$*, then* $e \rhd \underline{e'}$*, and (3) if* $X \mapsto \underline{e}$ *then* $e \in X$*.*

**Proposition C.17.**

1. *Given a CREBES,* $\mathcal{E} = (E, F, \mapsto, \rhd)$ *and corresponding CS* $C_{er}(\mathcal{E}) = (E, F, \mathsf{C}, \to)$*, any reachable* $C \in \mathsf{C}$ *is forwards-reachable.*
2. *If* $\mathcal{E} = (E, F, \mapsto, \sharp, \rhd)$ *is a CREBES and* $C_{br}(\mathcal{E}) = (E, F, \mathsf{C}, \to)$ *then whenever* $X \in \mathsf{C}$*,* $X \xrightarrow{A \cup \underline{B}} Y$ *and* $A \cup B \subseteq F$*, we get a transition* $Y \xrightarrow{B \cup \underline{A}} X$*.*

*Proof.* Similar to the proof of Proposition 3.8

**Definition C.18 (Labelled Reversible Bundle Event Structure (LREBES)).** *A labelled reversible extended bundle event structure* $\mathcal{E} = (E, F, \mapsto, \rhd, \lambda, \mathsf{Act})$ *consist of an REBES* $(E, F, \mapsto, \rhd)$*, a set of labels* $\mathsf{Act}$*, and a surjective labelling function* $\lambda : E \to \mathsf{Act}$*.*

**Definition C.19 (LRBES-morphism).** *Given LREBESs* $\mathcal{E}_0 = (E_0, F_0, \mapsto_0, \sharp_0, \rhd_0, \lambda_0, \mathsf{Act}_0)$ *and* $\mathcal{E}_1 = (E_1, F_1, \mapsto_1, \sharp_1, \rhd_1, \lambda_1, \mathsf{Act}_1)$*, a LREBES morphism* $f : \mathcal{E}_0 \to \mathcal{E}_1$ *is a partial function* $f : E_0 \to E_1$ *such that* $f : (E_0, F_0, \mapsto_0, \rhd_0) \to (E_1, F_1, \mapsto_1, \rhd_1)$ *is an REBES-morphism and for all* $e \in E_0$*, either* $f(e) = \bot$ *or* $\lambda_0(e) = \lambda_1(f(e))$*.*

# D Section 6

## D.1 Propagation Rules

## D.2 Lemma D.1

**Lemma D.1.** *Let* $P$ *be a consistent process with no subprocess* $\mathsf{rolling}\ \gamma$ *and let* $C$ *be a set of tags such that if* $\gamma \in C$ *and* $\gamma \leq_P \gamma'$ *then* $\gamma' \in C$*. Then* $P_{\lfloor C} \to^* P$*.*

*Proof.* We prove this by induction on the size of $C$.

Suppose $C = \emptyset$. Then $P_{\lfloor C} = P$.

Suppose $P_{\lfloor C'} \to^* P$ and $C = C' \cup \{\gamma\}$ for some $\gamma$ such that if $\gamma' \leq_P \gamma$ then $\gamma' \notin C'$. Then if there does not exist an action $\alpha$ and key $n$ such that $\alpha_\gamma[n]$ occurs in $P$, then $P_{\lfloor C} = P_{\lfloor C'} \to^* P$. If there exists a process $P'$, an action $\alpha$ and key $n$ such that $\alpha_\gamma[n].P'$ is a subprocess of $P$ then all past actions of $P'$ are in $C'$, meaning $P'_{\lfloor C} = \mathsf{rt}(P')$, and since $\gamma' \leq_P \gamma \Rightarrow \gamma' \notin C'$, we get $P_{\lfloor C} \xrightarrow{\alpha_\gamma[n]} P_{\lfloor C'} \to^* P$.

## D.3 Proof of Theorem 6.6

*Proof.* Follows from Lemma D.1.

(bound) $\dfrac{P \xrightarrow{\mu[m]} P'}{(\nu\ \gamma)P \xrightarrow{\mu[m]} (\nu\ \gamma)P'}$

(prop ROLL 1) $\dfrac{P \overset{\text{roll } \gamma}{\rightsquigarrow} P' \quad \gamma \neq \gamma'}{\beta_{\gamma'}[m].P \overset{\text{roll } \gamma}{\rightsquigarrow} \beta_{\gamma'}[m].P'}$

(prop ROLL 2) $\dfrac{P \overset{\text{roll } \gamma}{\rightsquigarrow} P'}{\beta_{\gamma'}.P \overset{\text{roll } \gamma}{\rightsquigarrow} \beta.\gamma'.P'}$

(prop ROLL 3) $\dfrac{P_0 \overset{\text{roll } \gamma}{\rightsquigarrow} P_0'}{P_0 + P_1 \overset{\text{roll } \gamma}{\rightsquigarrow} P_0' + P_1}$

(prop ROLL 4) $\dfrac{P \overset{\text{roll } \gamma}{\rightsquigarrow} P'}{P \setminus A \overset{\text{roll } \gamma}{\rightsquigarrow} P' \setminus A}$

(prop ROLL 5) $\dfrac{P \overset{\text{roll } \gamma}{\rightsquigarrow} P'}{P[f] \overset{\text{roll } \gamma}{\rightsquigarrow} P'[f]}$

(prop ROLL start 1) $\dfrac{P \overset{\text{start roll } \gamma}{\rightsquigarrow} P'}{\alpha_{\gamma'}[n].P \overset{\text{start roll } \gamma}{\rightsquigarrow} \alpha_{\gamma'}[n].P'}$

(prop ROLL start 2) $\dfrac{P_0 \overset{\text{start roll } \gamma}{\rightsquigarrow} P_0'}{P_0 \mid P_1 \overset{\text{start roll } \gamma}{\rightsquigarrow} P_0' \mid P_1}$

(prop ROLL start 3) $\dfrac{P \overset{\text{start roll } \gamma}{\rightsquigarrow} P'}{P \setminus A \overset{\text{start roll } \gamma}{\rightsquigarrow} P' \setminus A}$

(prop ROLL start 4) $\dfrac{P \overset{\text{start roll } \gamma}{\rightsquigarrow} P'}{P[f] \overset{\text{start roll } \gamma}{\rightsquigarrow} P'[f]}$

(prop ROLL start 5) $\dfrac{P \overset{\text{start roll } \gamma'}{\rightsquigarrow} P'}{(\nu\gamma)P \overset{\text{start roll } \gamma'}{\rightsquigarrow} (\nu\gamma)P'}$

(prop ROLL start 6) $\dfrac{P \equiv Q \overset{\text{start roll } \gamma'}{\rightsquigarrow} Q' \equiv P'}{P \overset{\text{start roll } \gamma'}{\rightsquigarrow} P'}$

(prop ROLL start 7) $\dfrac{P_0 \overset{\text{start roll } \gamma}{\rightsquigarrow} P_0' \quad \mathsf{std}(P_1) \quad \text{rolling } \gamma' \text{ is not a subprocess of } P_1}{P_0 + P_1 \overset{\text{start roll } \gamma}{\rightsquigarrow} P_0' + P_1}$

(prop ROLL 6) $\dfrac{P \overset{\text{roll } \gamma'}{\rightsquigarrow} P' \quad \gamma' \neq \gamma}{(\nu\ \gamma)P \overset{\text{roll } \gamma'}{\rightsquigarrow} (\nu\ \gamma)P'}$

(prop ROLL 7) $\dfrac{P \equiv Q \overset{\text{roll } \gamma'}{\rightsquigarrow} Q' \equiv P'}{P \overset{\text{roll } \gamma'}{\rightsquigarrow} P'}$

**Table 5.** The operational semantics for propagation of rolls and bound tags in Roll-CCSK

### D.4 Proof of Theorem 6.9

*Proof.* To get $P''$ we apply first $\overset{\text{start roll } \gamma_i}{\rightsquigarrow}$ for every $m_i \in T$ and then $\overset{\text{roll bound}}{\rightsquigarrow}$ or $\overset{\text{roll } \gamma_i}{\rightsquigarrow}$ for every $m_i \in T$ to $P$. We show that this is the correct $P''$.

Let $P_r$ be $P$ with every rolling $\gamma$ replaced with roll $\gamma$. Then $P_r \rightsquigarrow^* P_r''$, where $P_r''$ is $P''$ with every rolling $\gamma$ replaced with roll $\gamma$ using the same rules as $P \rightsquigarrow^* P''$.

By the loop theorem we get $P_r'' \to^* P_r$. And since $\phi(P_r) = \phi(P)$ and $\phi(P_r'') = \phi(P'')$, we can translate this computation into CCSK: $\phi(P'') \to^*_{CCSK} \phi(P)$. From the loop lemma of CCSK, this gives us $\phi(P) \rightsquigarrow^*_{CCSK} \phi(P'')$. And obviously $\phi(P'') \not\rightsquigarrow_T$.

We then only need to show that if $\phi(P) \rightsquigarrow^*_T P' \not\rightsquigarrow_T$, then $P' = \phi(P'')$. We then only need to show that if $\phi(P) \rightsquigarrow^*_T P' \not\rightsquigarrow_T$, then $P' = \phi(P'')$. Since they both reverse all the keys causally dependent on keys in $T$, this follows from proposition 5.16 of [19].

## E Section 7

### E.1 Proof of Proposition 7.3

We use the following lemmas:

**Lemma E.1.** *Let $P_0$ and $P_1$ be consistent processes such that $\{\!|P_0|\!\} = \langle \mathcal{E}_0, \mathsf{Init}_0, k_0 \rangle$, $\{\!|P_1|\!\} = \langle \mathcal{E}_1, \mathsf{Init}_1, k_1 \rangle$, and $\mathcal{E}_0 \leq \mathcal{E}_1$. If there exists $A\left\langle \tilde{b}, \tilde{\delta} \right\rangle$ in $P_0$ such that $A(\tilde{a}, \tilde{\gamma}) = P_A$ and $P_1 = P_0\{{}^{A\langle \tilde{b}, \tilde{\delta} \rangle}/_{(\nu\ \tilde{\delta}) P_A\{{}^{\tilde{a},\tilde{\gamma}}/_{\tilde{b},\tilde{\delta}}\}}\}$, and an action $\alpha_\gamma$ such that $\left\{\!\left|\alpha_\gamma.P_0\right|\!\right\} = \left\langle \mathcal{E}_0', \mathsf{Init}_0', k_0' \right\rangle$ and $\left\{\!\left|\alpha_\gamma.P_1\right|\!\right\} = \left\langle \mathcal{E}_1', \mathsf{Init}_1', k_1' \right\rangle$, then $\mathcal{E}_0' \leq \mathcal{E}_1'$.*

*Proof.* Obviously $E_0' \subseteq E_1'$ and $F_0' = F_1' \cap E_0$

If $X \mapsto_0' e$ then either $X \mapsto_0 e$, or $X = \{e_\alpha\}$, $e \in E_0$, and $\lambda_0(e) \neq \mathsf{roll}\ \gamma'$.

If $X \mapsto_0 e$ then there exists some $X_1 \subseteq E_1$ such that $X_1 \cap E_0 = X_0$ and $X_1 \mapsto_1 e$, meaning $X_1 \mapsto_1' e$ and $X_1 \cap E_0' = X_0$.

If $X = \{e_\alpha\}$, $e \in E_0$, and $\lambda_0(e) \neq \mathsf{roll}\ \gamma'$ then $e \in E_1$ and $\lambda_1(e) = \lambda_0(e) \neq \mathsf{roll}\ \gamma'$, meaning $\{e_\alpha\} \mapsto_1' e$.

If $X \mapsto_1' e$ and $e \in E_0'$ then either $X \mapsto_1 e$, or $X = \{e_\alpha\}$, $e \in E_1$, and $\lambda_1(e) \neq \mathsf{roll}\ \gamma'$.

If $X \mapsto_1 e$ and $e \in E_0'$ then $X \cap E_0 = X \cap E_0' \mapsto_0 e$, meaning $X \cap E_0' \mapsto_0 e$.

If $e \in E_0'$, $X = \{e_\alpha\}$, $e \in E_1$, and $\lambda_1(e) \neq \mathsf{roll}\ \gamma'$, then $\lambda_0(e) = \lambda_1(e) \neq \mathsf{roll}\ \gamma$, meaning $\{e_\alpha\} \mapsto_0' e$.

If $X \mapsto_0' \underline{e}$ then either $X = \{e\}$, or $e = e_\alpha$ and $X = \{e' \in E_0 \mid \lambda_0(e') = \mathsf{roll}\ \gamma\}$, or $\lambda_0(e) \in \{\mathsf{roll}\ \gamma', \mathsf{roll\ bound}\} \cup \{\mathsf{start\ roll}\ \gamma' \mid \nexists \beta, n.\beta_{\gamma'}$ or $\beta_{\gamma'}[n]$ occurs in $\alpha_\gamma.P_0\}$ and $X \mapsto_0 \underline{e}$, or $\lambda_0(e) \notin \{\mathsf{roll}\ \gamma', \mathsf{roll\ bound}\} \cup \{\mathsf{start\ roll}\ \gamma' \mid \nexists \beta, n.\beta_{\gamma'}$ or $\beta_{\gamma'}[n]$ occurs in $\alpha_\gamma.P_0\}$, $\{e\} \neq X' \mapsto_0 \underline{e}$, and $X = X' \cup \{e' \mid \lambda_0(e') = \mathsf{roll}\ \gamma\}$.

If $X = \{e\}$ then obviously $X \mapsto_1' e$.

If $e = e_\alpha$ and $X = \{e' \in E_0 \mid \lambda_0(e') = \mathsf{roll}\ \gamma\}$ then $X = \{e' \in E_1 \mid \lambda_1(e') = \mathsf{roll}\ \gamma\} \cap E_0$ and $\{e' \in E_1 \mid \lambda_1(e') = \mathsf{roll}\ \gamma\} \mapsto_1' e$.

If $\lambda_0(e) \in \{\mathsf{roll}\ \gamma', \mathsf{roll\ bound}\} \cup \{\mathsf{start\ roll}\ \gamma' \mid \nexists \beta, n.\beta_{\gamma'}$ or $\beta_{\gamma'}[n]$ occurs in $\alpha_\gamma.P_0\}$ and $X \mapsto_0 \underline{e}$ then $\lambda_1(e) \in \{\mathsf{roll}\ \gamma', \mathsf{roll\ bound}\} \cup \{\mathsf{start\ roll}\ \gamma' \mid \nexists \beta, n.\beta_{\gamma'}$ or $\beta_{\gamma'}[n]$ occurs in $\alpha_\gamma.P_1\}$ and there exists $X_1$ such that $X_1 \cap E_0 = X$ and $X_1 \mapsto_1 \underline{e}$, meaning $X_1 \mapsto_1' \underline{e}$.

If $\lambda_0(e) \notin \{\mathsf{roll}\ \gamma', \mathsf{roll\ bound}\} \cup \{\mathsf{start\ roll}\ \gamma' \mid \nexists \beta, n.\beta_{\gamma'}$ or $\beta_{\gamma'}[n]$ occurs in $\alpha_\gamma.P_0\}$, $\{e\} \neq X' \mapsto_0 \underline{e}$, and $X = X' \cup \{e' \mid \lambda_0(e') = \mathsf{roll}\ \gamma\}$ then there exists $X'_1 \subseteq E_1$ such that $X'_1 \cap E_0 = X'$ and $X'_1 \mapsto_1 e$. This means $X'_1 \cup \{e' \in E_1 \mid \lambda_1(e') = \mathsf{roll}\ \gamma\} \mapsto'_1 e$, and clearly $\{e' \in E_1 \mid \lambda_1(e') = \mathsf{roll}\ \gamma\} \cap E_0 = \{e' \in E_0 \mid \lambda_0(e') = \mathsf{roll}\ \gamma\}$, meaning $(X'_1 \cup \{e' \in E_1 \mid \lambda_1(e') = \mathsf{roll}\ \gamma\}) \cap E'_0 = X$.

If $X \mapsto'_1 \underline{e}$ and $e \in E'_0$ then either $X = \{e\}$, or $e = e_\alpha$ and $X = \{e' \in E_1 \mid \lambda_1(e') = \mathsf{roll}\ \gamma\}$, or $\lambda_1(e) \in \{\mathsf{roll}\ \gamma', \mathsf{roll\ bound}\} \cup \{\mathsf{start\ roll}\ \gamma' \mid \nexists \beta, n.\beta_{\gamma'}$ or $\beta_{\gamma'}[n]$ occurs in $\alpha_\gamma.P_1\}$ and $X \mapsto_1 \underline{e}$, or $\lambda_1(e) \notin \{\mathsf{roll}\ \gamma', \mathsf{roll\ bound}\} \cup \{\mathsf{start\ roll}\ \gamma' \mid \nexists \beta, n.\beta_{\gamma'}$ or $\beta_{\gamma'}[n]$ occurs in $\alpha_\gamma.P_1\}$, $\{e\} \neq X' \mapsto_1 \underline{e}$, and $X = X' \cup \{e' \mid \lambda_1(e') = \mathsf{roll}\ \gamma\}$.

If $X = \{e\}$ then obviously $X \mapsto'_0 e$.

If $e = e_\alpha$ and $X = \{e' \in E_1 \mid \lambda_1(e') = \mathsf{roll}\ \gamma\}$ then $\{e' \in E_0 \mid \lambda_0(e') = \mathsf{roll}\ \gamma\} \mapsto'_0 \underline{e}$ and obviously $X \cap E'_0 = \{e' \in E_0 \mid \lambda_0(e') = \mathsf{roll}\ \gamma\}$.

If $\lambda_1(e) \in \{\mathsf{roll}\ \gamma', \mathsf{roll\ bound}\} \cup \{\mathsf{start\ roll}\ \gamma' \mid \nexists \beta, n.\beta_{\gamma'}$ or $\beta_{\gamma'}[n]$ occurs in $\alpha_\gamma.P_1\}$ and $X \mapsto_1 \underline{e}$ then $\lambda_0(e) \in \{\mathsf{roll}\ \gamma', \mathsf{roll\ bound}\} \cup \{\mathsf{start\ roll}\ \gamma' \mid \nexists \beta, n.\beta_{\gamma'}$ or $\beta_{\gamma'}[n]$ occurs in $\alpha_\gamma.P_0\}$, and $X \cap E_0 \mapsto_0 \underline{e}$, meaning $X \cap E_0 \mapsto'_0 \underline{e}$.

If $\lambda_1(e) \notin \{\mathsf{roll}\ \gamma', \mathsf{roll\ bound}\} \cup \{\mathsf{start\ roll}\ \gamma' \mid \nexists \beta, n.\beta_{\gamma'}$ or $\beta_{\gamma'}[n]$ occurs in $\alpha_\gamma.P_1\}$, $\{e\} \neq X' \mapsto_1 \underline{e}$, and $X = X' \cup \{e' \mid \lambda_1(e') = \mathsf{roll}\ \gamma\}$ then $X' \cap E_0 \mapsto_0 \underline{e}$, meaning $(X' \cap E_0) \cup \{e' \mid \lambda_0(e') = \mathsf{roll}\ \gamma\} \mapsto'_0 \underline{e}$, and obviously $X \cap E_0 = (X' \cap E_0) \cup \{e' \mid \lambda_0(e') = \mathsf{roll}\ \gamma\}$.

If $e \rhd'_0 e'^*$ then either $\lambda_0(e) \in \{\mathsf{roll}\ \gamma', \mathsf{roll\ bound}\} \cup \{\mathsf{start\ roll}\ \gamma' \mid \nexists \beta, n.\beta_{\gamma'}$ or $\beta_{\gamma'}[n]$ occurs in $\alpha_\gamma.P_0\}$ and $e'^* = \underline{e_\alpha}$, or $e = e_\alpha$, $e'^* = \underline{e'}$ and $\lambda_0(e') = \mathsf{roll}\ \gamma$, or $\lambda_0(e) = \mathsf{roll}\ \gamma$ and $e'^* = e_\alpha$, or $\lambda_0(e) = \mathsf{roll}\ \gamma$, $e'^* = e'$, and $\lambda_0(e') \notin \{\mathsf{roll}\ \gamma', \mathsf{roll\ bound}\} \cup \{\mathsf{start\ roll}\ \gamma' \mid \nexists \beta, n.\beta_{\gamma'}$ or $\beta_{\gamma'}[n]$ occurs in $\alpha_\gamma.P_0\}$.

If $\lambda_0(e) \in \{\mathsf{roll}\ \gamma', \mathsf{roll\ bound}\} \cup \{\mathsf{start\ roll}\ \gamma' \mid \nexists \beta, n.\beta_{\gamma'}$ or $\beta_{\gamma'}[n]$ occurs in $\alpha_\gamma.P_0\}$ and $e'^* = \underline{e_\alpha}$, then $\lambda_1(e) \in \{\mathsf{roll}\ \gamma', \mathsf{roll\ bound}\} \cup \{\mathsf{start\ roll}\ \gamma' \mid \nexists \beta, n.\beta_{\gamma'}$ or $\beta_{\gamma'}[n]$ occurs in $\alpha_\gamma.P_1\}$, and therefore $e \rhd'_1 \underline{e_\alpha}$.

If $e = e_\alpha$, $e'^* = \underline{e'}$ and $\lambda_0(e') = \mathsf{roll}\ \gamma$ then $\lambda_1(e') = \mathsf{roll}\ \gamma$, and therefore $e_\alpha \rhd'_1 e'$.

If $\lambda_0(e) = \mathsf{roll}\ \gamma$ and $e'^* = e_\alpha$, then $\lambda_1(e) = \mathsf{roll}\ \gamma$, and therefore $e \rhd'_1 e_\alpha$.

If $\lambda_0(e) = \mathsf{roll}\ \gamma$, $e'^* = e'$, and $\lambda_0(e') \notin \{\mathsf{roll}\ \gamma', \mathsf{roll\ bound}\} \cup \{\mathsf{start\ roll}\ \gamma' \mid \nexists \beta, n.\beta_{\gamma'}$ or $\beta_{\gamma'}[n]$ occurs in $\alpha_\gamma.P_0\}$, then $\lambda_1(e) = \mathsf{roll}\ \gamma$ and $\lambda_1(e') \notin \{\mathsf{roll}\ \gamma', \mathsf{roll\ bound}\} \cup \{\mathsf{start\ roll}\ \gamma' \mid \nexists \beta, n.\beta_{\gamma'}$ or $\beta_{\gamma'}[n]$ occurs in $\alpha_\gamma.P_0\}$, meaning $e \rhd'_1 e'$.

If $e \rhd'_1 e'^*$ for $e, e' \in E_0$ then the argument is similar.

Obviously $\lambda'_0 = \lambda'_1 \restriction_{E'_0}$ and $\mathsf{Act}$ is the range of $\lambda'_0$.

**Lemma E.2.** *Let $P_0$ and $P_1$ be consistent processes such that $\{\!|P_0|\!\} = \langle \mathcal{E}_0, \mathsf{Init}_0, k_0 \rangle$, $\{\!|P_1|\!\} = \langle \mathcal{E}_1, \mathsf{Init}_1, k_1 \rangle$, and $\mathcal{E}_0 \leq \mathcal{E}_1$. If there exists $A\left\langle \tilde{b}, \tilde{\delta} \right\rangle$ in $P_0$ such that $A(\tilde{a}, \tilde{\gamma}) = P_A$ and $P_1 = P_0\{^{A\langle \tilde{b}, \tilde{\delta} \rangle}/_{(\nu\ \tilde{\delta})P_A\{^{\tilde{b}, \tilde{\delta}}/_{\tilde{a}, \tilde{\gamma}}\}}\}$, and an action $\alpha_\gamma$ such that $\left\{\!\left|\alpha_\gamma[m].P_0\right|\!\right\} = \left\langle \mathcal{E}'_0, \mathsf{Init}'_0, k'_0 \right\rangle$ and $\left\{\!\left|\alpha_\gamma[m].P_1\right|\!\right\} = \left\langle \mathcal{E}'_1, \mathsf{Init}'_1, k'_1 \right\rangle$, we get $\mathcal{E}'_0 \leq \mathcal{E}'_1$.*

*Proof.* Follows from Lemma E.1 and the definitions of $\left\{\!\left|\alpha_\gamma[m].P_0\right|\!\right\}$ and $\left\{\!\left|\alpha_\gamma[m].P_1\right|\!\right\}$

**Lemma E.3.** *Let $P_0 \mid P_2$, $P_1 \mid P_2$ be consistent processes such that $\{\!|P_0|\!\} = \langle \mathcal{E}_0, \mathsf{Init}_0, k_0 \rangle$, $\{\!|P_1|\!\} = \langle \mathcal{E}_1, \mathsf{Init}_1, k_1 \rangle$, $\{\!|P_0 \mid P_2|\!\} = \left\langle \mathcal{E}'_0, \mathsf{Init}'_0, k'_0 \right\rangle$, $\{\!|P_1 \mid P_2|\!\} = \left\langle \mathcal{E}'_1, \mathsf{Init}'_1, k'_1 \right\rangle$, and $\mathcal{E}_0 \leq \mathcal{E}_1$. Then $\mathcal{E}'_0 \leq \mathcal{E}'_1$.*

*Proof.* For all $e \in E'_0$ we know either $e \in E_0 \times_* E_2$ and $\lambda'_0(e) \in \{\mathsf{roll}\ \gamma, \mathsf{start\ roll}\ \gamma, \mathsf{roll\ bound},$ or $e = (X, e')$ for some $e' \in E_0 \times_* E_2$ and $X \in \mathsf{causes}(e')$. If $e \in E_0 \times_* E_2$ and $\lambda'_0(e) \in \{\mathsf{roll}\ \gamma, \mathsf{start\ roll}\ \gamma, \mathsf{roll\ bound}$ then obviously $e \in E'_1$. If $e = (X, e')$ then if $e' = (e_0, *)$ then for each $(e'_0, e'_2) \in X$ there exists $X_0$ such that $e'_0 \in X_0$ and $X_0 \mapsto_0 e_0$. This means there exists $X_1 \subseteq E_1$ such that $X_1 \mapsto_1 e_0$ and $X_0 = X_1 \cap E_0$. In addition, for any $X'_1 \subseteq E_1$ such that $X'_1 \mapsto_1 e_0$, we have $X'_1 \cap E_0 \mapsto_0 e_0$, and therefore $(X'_1 \times E_2) \cap X \neq \emptyset$. We therefore get $e \in E'_1$. If $e' = (*, e_2)$ then obviously $e_2$'s causes are the same in $\mathcal{E}'_1$ and therefore $e' \in E'_1$. If $e' = (e_0, e_2)$ then the argument is a combination of the first two cases.

Obviously $F'_0 = E'_0 \cap E'_1$.

If $X \mapsto'_0 e$ then either $e = (X', e')$, $X = \{(X'', e'') \mid X'' \subseteq X'\}$, and $e'' \in X'$, or $e = (e_0, e_2)$ and there exists $X'$ such that $X' \mapsto_{0\times 2} e$ and $X = \{e' \mid (\pi_0(e'), \pi_2(e')) \in X'\}$.

If $e = (X', e')$, $X = \{(X'', e'') \mid X'' \subseteq X'\}$, and $e'' \in X'$ then clearly $X \mapsto'_1 e$.

If $e = (e_0, e_2)$ and there exists $X'$ such that $X' \mapsto_{0\times 2} e$ and $X = \{e' \mid (\pi_0(e'), \pi_2(e')) \in X'\}$ then $X' \mapsto_{1\times 2} e$, and obviously $\{e' \mid (\pi_0(e'), \pi_2(e')) \in X'\} = \{e' \mid (\pi_1(e'), \pi_2(e')) \in X'\} \cap E'_0$.

If $X \mapsto'_1 e$ and $e \in E'_0$ then either $e = (X', e')$, $X = \{(X'', e'') \mid X'' \subseteq X'\}$, and $e'' \in X'$, or $e = (e_1, e_2)$ and there exists $X'$ such that $X' \mapsto_{1\times 2} e$ and $X = \{e' \mid (\pi_1(e'), \pi_2(e')) \in X'\}$.

If $e = (X', e')$, $X = \{(X'', e'') \mid X'' \subseteq X'\}$, and $e'' \in X'$ then since $e \in E'_0$, for all $e''' \in X'$, $\pi_1(e''') \in E_0$, meaning $X \subseteq E'_0$, and $X \mapsto'_0 e$.

If $e = (e_1, e_2)$ and there exists $X'$ such that $X' \mapsto_{1\times 2} e$ and $X = \{e' \mid (\pi_1(e'), \pi_2(e')) \in X'\}$ then $e_1 \in E_0 \cup \{*\}$, and $X' \cap (E_0 \times_* E_2) \mapsto_{0\times 2} e$, meaning $X \cap E'_0 = \{e' \mid (\pi_0(e'), \pi_2(e')) \in X' \cap (E_0 \times_* E_2)\} \mapsto e$.

If $X \mapsto'_0 \underline{e}$ then either $X = \{e\}$, or $e = (X', (e_0, e_2))$ and $X = \bigcup \left\{ X'' \;\middle|\; \begin{array}{l} \exists i \in \{0,2\}, X_i \in E_i . X_i \mapsto \underline{\pi_i(e)} \\ \text{or } \exists e_\times \in X' . X_i \mapsto \underline{\pi_i(e_\times)} \\ \text{, and } e' \in X'' \text{ iff } \underline{\pi_i(e') \in X_i} \end{array} \right\}$, or $e = (e_0, e_1)$ and there exists $X'$ such that $X' \mapsto_{\times_0} e$ and $X = \{e' \mid (\pi_0(e'), \pi_1(e')) \in X'\}$.

If $X = \{e\}$ then obviously $X \mapsto'_1 \underline{e}$.

If $e = (X', (e_0, e_2))$ and $X = \left\{ X'' \;\middle|\; \begin{array}{l} \exists i \in \{0,2\}, X_i \in E_i . X_i \mapsto_i \underline{\pi_i(e)} \\ \text{or } \exists e_\times \in X' . X_i \mapsto_i \underline{\pi_i(e_\times)} \\ \text{, and } e' \in X'' \text{ iff } \underline{\pi_i(e') \in X_i} \end{array} \right\}$ then (1) for each $X_0$ such that $X_0 \mapsto_0 \underline{e_0}$, there exists $X_1$ such that $X_1 \mapsto_1 \underline{e_0}$ and $X_1 \cap E_0 = X_0$, and (2) for each $X_0$ such that there exists $(e'_0, e'_2) \in X'$, such that $X_0 \mapsto'_0 \underline{e_0}$, there exists $X_1$ such that $X_1 \mapsto_1 \underline{e'_0}$ and $X_1 \cap E_0 = X_0$, meaning

$$\bigcup \left\{ X'' \;\middle|\; \begin{array}{l} \exists i \in \{1,2\}, X_i \in E_i . X_i \mapsto_i \underline{\pi_i(e)} \\ \text{or } \exists e_\times \in X' . X_i \mapsto_i \underline{\pi_i(e_\times)} \\ \text{, and } e' \in X'' \text{ iff } \underline{\pi_i(e') \in X_i} \end{array} \right\} \cap E_0 = X$$

If $e = (e_0, e_1)$ and there exists $X'$ such that $X' \mapsto_{0\times 2} e$ and $X = \{e' \mid (\pi_0(e'), \pi_1(e')) \in X'\}$, then since $||$ is monotonic, there exists $X''$ such that $X' = X'' \cap (E_0 \times_* E_2)$ and $X'' \mapsto_{1\times 2} e$, meaning $\{e' \mid (\pi_1(e'), \pi_1(e')) \in X''\} \mapsto'_1 e$.

If $X \mapsto'_1 \underline{e}$ and $e \in E'_0$ then $X = \{e\}$, or $e = (X', (e_1, e_2))$ and $X = \bigcup\{X'' \mid \exists i \in \{0,1\}.\pi_i(X'') \mapsto \underline{e_i}$ or $\exists e' \in X'.\pi_i(X'') \mapsto_i \pi_i(\underline{e'})\}$. If $X = \{e\}$ then obviously $X \mapsto'_0 \underline{e}$. If $e = (X', (e_1, e_2))$ and $X = \bigcup \left\{ X'' \;\middle|\; \begin{array}{l} \exists i \in \{1,2\}, X_i \in E_i.X_i \mapsto_i \underline{\pi_i(e)} \\ \text{or } \exists e_\times \in X'.X_i \mapsto_i \underline{\pi_i(e_\times)} \\ \text{, and } e' \in X'' \text{ iff } \pi_i(e') \in X_i \end{array} \right\}$ then for each $X_1$ such that $X_1 \mapsto \underline{e_1}$, we know $X_1 \cap E_0 \mapsto \underline{e_1}$, and for each $X_1$ such that there exists $(e'_1, e'_2) \in X'$ such that $X_1 \mapsto'_1 \underline{e_1}$, since $e \in E'_0$, $e'_1 \in E_0$, meaning $X_1 \cap E_0 \mapsto e_1$. Therefore $X \cap E'_0 \mapsto'_0 e_1$.

If $e \rhd'_0 e'^*$ then there exists $i \in \{0,2\}$ such that either (1) $\pi_i(e) \rhd_i \pi_i(e')^*$, or (2) $\pi_i(e) = \pi_i(e') \neq \bot$, or (3) $e'^* = e'$, $e \neq e'$, and $e \in X \mapsto \underline{e'}$, or (4) and there exist $\gamma$, $\gamma'$ such that $\lambda(e) = \mathsf{roll}\ \gamma$ and $\lambda(e') = \mathsf{roll}\ \gamma'$. In all these cases it is clear that the same conditions will apply in $\mathcal{E}'_1$.

Similar logic applies if $e \rhd'_1 e'^*$ and $e, e' \in E'_0$.

Obviously $\lambda'_0 = \lambda'_1 \restriction_{E'_0}$ and $\mathsf{Act}$ is the range of $\lambda'_0$.

**Lemma E.4.** *Let $P_0$ and $P_1$ be consistent processes such that $\{\!|P_0|\!\} = \langle \mathcal{E}_0, \mathsf{Init}_0, k_0\rangle$, $\{\!|P_1|\!\} = \langle \mathcal{E}_1, \mathsf{Init}_1, k_1\rangle$, and $\mathcal{E}_0 \leq \mathcal{E}_1$. If there exists a tag $\gamma$ such that $\{\!|(\nu\,\gamma)P_0|\!\} = \langle \mathcal{E}'_0, \mathsf{Init}'_0, k'_0\rangle$ and $\{\!|(\nu\,\gamma)P_1|\!\} = \langle \mathcal{E}'_1, \mathsf{Init}'_1, k'_1\rangle$, we get $\mathcal{E}'_0 \leq \mathcal{E}'_1$.*

*Proof.* Obvious

And having those lemmas makes the proof simple.

*Proof.* Follows from Lemmas E.1, E.2, E.3, and E.4 and Proposition 4.7.

### E.2 Lemma E.5

**Lemma E.5 (Causal consistency of actions).** *Let $P$ be a process and $\{\!|P|\!\} = \langle \mathcal{E}, \mathsf{Init}, k\rangle$. Then for any events $e, e'$, if $e \in X \mapsto e'$ and $e' \not\rhd \underline{e}$, then there exists $\gamma$ such that $\lambda(e') \in \{\mathsf{roll}\ \gamma, \mathsf{roll\ bound}, \mathsf{start\ roll}\ \gamma\}$.*

*Proof.* We prove this by structural induction in $P$.
Suppose $P = 0$. Then there are no events and the lemma is trivially true.
Suppose $P = \mathsf{roll}\ \gamma$. Then $e \in X \mapsto e'$ means $e = e_t$ and $e' = e_r$, and obviously $\lambda(e') = \mathsf{roll}\ \gamma$.
Suppose $P = \mathsf{rolling}\ \gamma$. Then the argument is the same as the previous case.
Suppose $P = \alpha_{\gamma'}.P'$. Then $\{\!|P'|\!\} = \langle \mathcal{E}', \mathsf{Init}, k\rangle$ and either $X \mapsto' e'$ or $X = \{e_\alpha\}$. If $X \mapsto' e'$ then $e' \neq e_\alpha$ and by induction if $e' \not\rhd' \underline{e}$ then $\lambda(e') \in \{\mathsf{roll}\ \gamma, \mathsf{roll\ bound}, \mathsf{start\ roll}\ \gamma\}$. If $X = \{e_\alpha\}$ then, $e' \rhd e_\alpha$ unless $\lambda(e') \in \{\mathsf{roll}\ \gamma, \mathsf{roll\ bound}, \mathsf{start\ roll}\ \gamma\}$.
Suppose $P = \alpha_{\gamma'}.[m]P'$. Then the argument is the same as the previous case.
Suppose $P = P_0 + P_1$. Then $e = (i, e_i)$, $e' = (i, e'_i)$, $e_i \mapsto_i e'_i$, and $e'_i \not\rhd_i \underline{e_i}$, meaning $\lambda_i(e'_i) \in \{\mathsf{roll}\ \gamma, \mathsf{roll\ bound}, \mathsf{start\ roll}\ \gamma\}$, and therefore $\lambda(e') \in \{\mathsf{roll}\ \gamma, \mathsf{roll\ bound}, \mathsf{start\ roll}\ \gamma\}$.

Suppose $P = P_0 \mid P_1$. Then if $e' = (Y', e'')$, $e = (Y, e''')$ and $e''' \in Y'$, meaning there exists $i \in \{0, 1\}$ such that $\pi_i(e) \in X_i \mapsto_i \pi_i(e')$. By induction we get that, if $e'_i \not\bowtie_i \underline{e_i}$, then there exists $\gamma$ such that $\lambda_i(e'_i) \in \{\mathsf{roll}\ \gamma, \mathsf{roll\ bound}, \mathsf{start\ roll}\ \gamma\}$, meaning $e'_{1-i} =*$ and $\lambda(e') \in \{\mathsf{roll}\ \gamma, \mathsf{roll\ bound}, \mathsf{start\ roll}\ \gamma\}$, and if $e'_i \rhd_i \underline{e_i}$ then $e' \rhd \underline{e}$. If $e' \in E_\times$ then $\lambda(e') \in \{\mathsf{roll}\ \gamma, \mathsf{roll\ bound}, \mathsf{start\ roll}\ \gamma\}$
Suppose $P = P' \setminus A$. Then the lemma follows from induction.
Suppose $P = (\nu\ \gamma')P'$. Then the lemma follows from induction.
Suppose $P = A\left\langle \tilde{b}, \tilde{\delta} \right\rangle$. Then the lemma holds if it holds for $P_A$.

### E.3 Lemma E.6

**Lemma E.6 (Transitive causation).** *Let $P$ be a process and $\{\!|P|\!\} = \langle \mathcal{E}, \mathsf{Init}, k \rangle$. Then whenever $X \mapsto e \in X' \mapsto e'$, we have $X \mapsto e'$, or there exists a $\gamma$ such that $\lambda(e') \in \{\mathsf{roll}\ \gamma, \mathsf{roll\ bound}\}$.*

*Proof.* We prove this by structural induction on $P$.

- Suppose $P = 0$. Then there are no events and the lemma is trivially true.
- Suppose $P = \mathsf{roll}\ \gamma$. Then no $X, X', e, e'$ exist such that $X \mapsto e \in X' \mapsto e'$.
- Suppose $P = \mathsf{rolling}\ \gamma$. Then the argument is the same as the previous case.
- Suppose $P = \alpha_{\gamma'}.P'$. Then $\{\!|P'|\!\} = \langle \mathcal{E}', \mathsf{Init}, k \rangle$, $X' \mapsto' e'$, and either $X = \{e_\alpha\}$, or $X \mapsto' e$. If $X = \{e_\alpha\}$, then $X \mapsto e'$ whenever $\lambda(e') \neq \mathsf{roll}\ \gamma'$. If $X \mapsto' e$ and $X' \mapsto' e'$ then by induction, $X \mapsto e'$.
- Suppose $P = \alpha_{\gamma'}.[m]P'$. Then the argument is the same as the previous case.
- Suppose $P = P_0 + P_1$. Then there exists an $i \in \{0, 1\}$ such that $e = (i, e_i)$, $e' = (i, e'_i)$, $\{e''_i \mid (i, e''_i) \in X'\} \mapsto_i e'_i$, and $\{e''_i \mid (i, e''_i) \in X\} \mapsto_i e_i$, meaning by induction $\{e''_i \mid (i, e''_i) \in X\} \mapsto_i e'_i$, and therefore $X \mapsto e'$.
- Suppose $P = P_0 \mid P_1$. Then $e' = (Y', (e'_0, e'_1))$ or there exists a $\gamma$ such that $\lambda(e') \in \{\mathsf{roll}\ \gamma, \mathsf{roll\ bound}\}$. If $e' = (Y', (e'_0, e'_1))$ then $e = (Y, (e_0, e_1))$ and $(e_0, e_1) \in Y'$, meaning there exists $i \in \{0, 1\}$ such that $e_i \in X_i \mapsto_i e'_i$. Similarly, $X = \{(Y'', e'') \mid (Y'', e'') \in E\}$ for some $e'' \in Y$. Since $Y \in \mathsf{cause}(e)$ and $e \in Y' \in \mathsf{cause}(e')$, there exists $Y'' \in \mathsf{cause}(e')$ such that $Y \subseteq Y''$. This means $X \mapsto e$.
- Suppose $P = P' \setminus A$. Then the lemma obviously follows from induction.
- Suppose $P = (\nu\ \gamma')P'$. Then the lemma obviously follows from induction.
- Suppose $P = A\left\langle \tilde{b}, \tilde{\delta} \right\rangle$. Then the lemma holds if it holds for $P_A$.

### E.4 Lemma E.7

**Lemma E.7 (forwards bundles).** *Let $P$ be a process and $\{\!|P|\!\} = \langle \mathcal{E}, \mathsf{Init}, k \rangle$. Then whenever $X \mapsto e$, either there exists $e'$ such that $X = \{e'\}$, or there exists a $\gamma$ such that $\lambda(e) \in \{\mathsf{roll}\ \gamma, \mathsf{roll\ bound}\}$.*

*Proof.* We prove this be structural induction on $P$.

- Suppose $P = 0$. Then $E = \emptyset$ and the case is trivial.
- Suppose $P = \mathsf{roll}\ \gamma$. Then $e' = e_t$ and $e = e_r$.

- Suppose $P = \mathsf{rolling}\ \gamma$. Then $e' = e_t$ and $e = e_r$.
- Suppose $P = \alpha_\gamma.P'$. Then by induction, if $X \mapsto_{P'} e$, then there exists an $e'$ such that $X = \{e'\}$. If $X \not\mapsto_{P'} e$, then $X = \{e_\alpha\}$.
- Suppose $P = \alpha_\gamma[m].P'$. Then by induction, if $X \mapsto_{P'} e$, there exists an $e'$ such that $X = \{e'\}$. If $X \not\mapsto_{P'} e$, then $X = \{e_\alpha\}$.
- Suppose $P = P_0 + P_1$. Then, by induction if $e = (i, e_i)$ and $X_i \mapsto_i e_i$, then $X_i = e_i'$, and $e' = (i, e_i')$.
- Suppose $P = P_0 \mid P_1$. Then either $e = (Y, e_\times)$ or there exists a $\gamma$ such that $\lambda(e) \in \{\mathsf{roll}\ \gamma, \mathsf{roll\ bound}\}$. If $e = (Y, e_\times)$ then $X = \{(Y'', e'') \mid (Y'', e'') \in E\}$ for some $e'' \in Y$. We therefore need to show that given an event $e'' \in Y$, there exists exactly one $Y'' \in \mathsf{cause}(e'')$ such that $Y'' \subseteq Y$. This follows naturally from items 2 and 3 of Definition 7.1.
- Suppose $P = P' \setminus A$. Then the lemma obviously follows from the definition of $\rho$.
- Suppose $P = (\nu\gamma)P'$. Then the lemma obviously follows from induction.
- Suppose $P = A\left\langle \tilde{b}, \tilde{\delta} \right\rangle$. Then the lemma holds if it holds for $P_A$.

### E.5 Lemma E.8

**Lemma E.8 (Reverse inverse causality).** *Let $P$ be a process and $[\![ P ]\!] = \langle \mathcal{E}, \mathsf{Init}, k \rangle$. Then whenever $e' \in X \mapsto \underline{e}$ and $e \neq e'$, we get $e' \rhd e$.*

*Proof.* We prove this by structural induction on $P$:

- Suppose $P = 0$. Then $E = \emptyset$ and the case is trivial.
- Suppose $P = \mathsf{roll}\ \gamma$. Then $e' = e_r$ and $e = e_t$.
- Suppose $P = \mathsf{rolling}\ \gamma$. Then $e' = e_r$ and $e = e_t$.
- Suppose $P = \alpha_\gamma.P'$. Then either $e = e_\alpha$ and $X = \{e'' \mid \lambda'_{P'}(e'') = \mathsf{roll}\ \gamma\}$, or $\lambda_{P'}(e) \in \{\mathsf{roll}\ \gamma', \mathsf{roll\ bound}\}$ and $X \mapsto_{P'} \underline{e}$, or $\lambda'_{P'}(e'') \notin \{\mathsf{roll}\ \gamma', \mathsf{roll\ bound}\}$, $\{e\} \neq X' \mapsto_{P'} e$, and $X = X' \cup \{e'' \mid \lambda'_{P'}(e'') = \mathsf{roll}\ \gamma\}$.
  In either case, it is clear that $e' \rhd \underline{e}$.
- Suppose $P = \alpha_\gamma[m].P'$. Then the argument is similar to the previous case.
- Suppose $P = P_0 + P_1$. Then, by induction if $e = (i, e_i)$ and $e_i' \in X_i \mapsto_i \underline{e_i}$, then $e_i' \rhd \underline{e_i}$, and $e' = (i, e_i')$, meaning $e' \rhd \underline{e}$.
- Suppose $P = P_0 \mid P_1$. Then the lemma holds by definition.
- Suppose $P = P' \setminus A$. Then the lemma obviously follows from induction.
- Suppose $P = (\nu\gamma)P'$. Then the lemma obviously follows from induction.
- Suppose $P = A\left\langle \tilde{b}, \tilde{\delta} \right\rangle$. Then the lemma holds if it holds for $P_A$.

### E.6 Lemma E.9

**Lemma E.9 (Single backwards bundle).** *Given a process $P$ such that $[\![ P ]\!] = \langle \mathcal{E}, \mathsf{Init}, k \rangle$, for any event $e \in F$, there exists at most one bundle $X \mapsto \underline{e}$ such that $X \neq \{e\}$.*

*Proof.* We prove this by structural induction in $P$:

- Suppose $P = 0$. Then $E = \emptyset$ and the case is trivial.

- Suppose $P = \mathsf{roll}\ \gamma$. Then $X = \{e_r\}$ and $e = e_t$.
- Suppose $P = \mathsf{rolling}\ \gamma$. Then $X = \{e_r\}$ and $e = e_t$.
- Suppose $P = \alpha_\gamma.P'$. Then either $e = e_\alpha$ and $X = \{e'' \mid \lambda'_{P'}(e'') = \mathsf{roll}\ \gamma\}$, or $\lambda_{P'}(e) \in \{\mathsf{roll}\ \gamma', \mathsf{roll\ bound}\}$ and $X \mapsto_{P'} \underline{e}$, or $\lambda'_{P'}(e'') \notin \{\mathsf{roll}\ \gamma', \mathsf{roll\ bound}\}$, $\{e\} \neq X' \mapsto_{P'} e$, and $X = X' \cup \{e'' \mid \lambda'_{P'}(e'') = \mathsf{roll}\ \gamma\}$.
  In either case, it is clear that there exists only one $X$.
- Suppose $P = \alpha_\gamma[m].P'$. Then the argument is similar to the previous case.
- Suppose $P = P_0 + P_1$. Then, by induction if $e = (i, e_i)$ then there exists at most one $X_i$ such that $e'_i \in X_i \mapsto_i \underline{e_i}$, meaning $\{i\} \times X_i \mapsto e$.
- Suppose $P = P_0 \mid P_1$. Then either $e = (X', e')$, in which case the lemma obviously holds, or there exists $X'$ such that $X' \mapsto_\times e$ and $X = \{e' \mid (\pi_0(e'), \pi_1(e')) \in X'\}$. By induction, since there exists an $i \in \{0, 1\}$ such that $\pi_i(e) = \bot$, there can only exist one such $X'$.
- Suppose $P = P' \setminus A$. Then the lemma obviously follows from induction.
- Suppose $P = (\nu\gamma)P'$. Then the lemma obviously follows from induction.
- Suppose $P = A\left\langle \tilde{b}, \tilde{\delta} \right\rangle$. Then the lemma holds if it holds for $P_A$.

### E.7 Lemma E.10

**Lemma E.10 (Reverse transitivity).** *Let $P$ be a process and $[\![P]\!] = \langle \mathcal{E}, \mathsf{Init}, k \rangle$. Then whenever $e' \in X \mapsto e$, $X' \mapsto \underline{e'}$, $X' \neq \{e'\}$, and $\lambda(e) = \mu$, there must exist $X'' \supseteq X'$ such that $X'' \mapsto \underline{e}$.*

*Proof.* We prove this by structural induction in $P$:

- Suppose $P = 0$, then $E = \emptyset$ and the case is trivial.
- Suppose $P = \mathsf{roll}\ \gamma$. Then there does not exist any $e$ such that $\lambda(e) = \mu$.
- Suppose $P = \mathsf{rolling}\ \gamma$. Then there does not exist any $e$ such that $\lambda(e) = \mu$.
- Suppose $P = \alpha_\gamma.P'$. Then either $X \mapsto_{P'} e$ or $X = \{e_\alpha\}$ and $e \in E_{P'}$.
  If $X \mapsto_{P'} e$ then there exists an $X_{P'}$ such that $X_{P'} \mapsto_{P'} \underline{e}$ and $X' = X_{P'} \cup \{e'' \mid \lambda_{P'}(e'') = \mathsf{roll}\ \gamma\}$. This means there exists $X'_{P'}$ such that $X'_{P'} \mapsto_{P'} \underline{e'}$, $X_{P'} \subseteq X'_{P'}$, and $X'_{P'} \cup \{e'' \mid \lambda_{P'}(e'') = \mathsf{roll}\ \gamma\} = X'' \mapsto \mathit{underlinee}$.
  If $X = \{e_\alpha\}$ and $e \in E_{P'}$ then $X' = \{e'' \mid \lambda_{P'}(e'') = \mathsf{roll}\ \gamma\}$ and there exists an $X_{P'}$ such that $X_{P'} \mapsto_{P'} \underline{e}$ and $X'' = X_{P'} \cup \{e'' \mid \lambda_{P'}(e'') = \mathsf{roll}\ \gamma\}$, meaning clearly $X' \subseteq X''$.
- Suppose $P = \alpha_\gamma[m].P'$. Then the argument is similar to the previous case.
- Suppose $P = P_0 + P_1$. Then, if $e = (i, e_i)$ and $e' = (i, e'_i)$ then there exists $X'_i$ such that $X' = \{i\} \times X'_i$ and $X'_i \mapsto \underline{e'_i}$, meaning there exists $X''_i \supseteq X'_i$ such that $X''_i \mapsto \underline{e_i}$ and therefore $\{i\} \times X''_i = X'' \mapsto \underline{e}$.
- Suppose $P = P_0 \mid P_1$. Then either $e = (Y, e_\times)$, or there exists $X'$ such that $X' \mapsto_\times e$ and $X = \{e' \mid (\pi_0(e'), \pi_1(e')) \in X'\}$.
  If $e = (Y, e_\times)$ then $e' = (Y', e'_\times)$ and $Y' \cup \{e'_\times\} \subseteq Y$, and
  $$X' = \bigcup \left\{ X''' \;\middle|\; \begin{array}{l} \exists i \in \{0,1\}, X_i \in E_i.X_i \mapsto_i \underline{\pi_i(e')} \\ \text{ or } \exists e''_\times \in Y'.X_i \mapsto_i \underline{\pi_i(e''_\times)} \\ \text{, and } e'' \in X''' \text{ iff } \underline{\pi_i(e'')} \in X_i \end{array} \right\}.$$

We define $X'' = \bigcup \left\{ X''' \;\middle|\; \begin{array}{l} \exists i \in \{0,1\}, X_i \in E_i . X_i \mapsto_i \underline{\pi_i(e)} \\ \text{ or } \exists e''_\times \in Y . X_i \mapsto_i \underline{\pi_i(e''_\times)} \\ \text{, and } e'' \in X''' \text{ iff } \underline{\pi_i(e'')} \in X_i \end{array} \right\}$ and show that $X' \subseteq X''$. By definition, since $e_\times \in Y'$, whenever $X_i \mapsto \pi_i(e')$ we get $\pi(e'') \in X_i$ iff $e'' \in X''$. And if there exists $e''_\times \in Y'$ such that $X_i \mapsto \pi_i(e''_\times)$ then, since $Y' \subseteq Y$, $e''_\times \in Y$ and therefore $\pi(e'') \in X_i$ iff $e'' \in X''$.

- Suppose $P = P' \setminus A$. Then the lemma obviously follows from induction.
- Suppose $P = (\nu\gamma)P'$. Then the lemma obviously follows from induction.
- Suppose $P = A\left\langle \tilde{b}, \tilde{\delta} \right\rangle$. Then the lemma holds if it holds for $P_A$.

### E.8 Proof of Proposition 7.4

We use Lemmas E.5–E.10.

*Proof.* We say that $\mathcal{E} = (E, F, \mapsto, \rhd, \lambda, \mathsf{Act})$ and $\mathcal{E}' = (E', F', \mapsto', \rhd', \lambda', \mathsf{Act}')$ and do a case analysis on the Structural congruence rules:

**$P = Q \mid R$ and $P' = R \mid Q$:** Then there exist $\mathcal{E}_Q$ and $\mathcal{E}_R$ such that for $i \in \{Q, R\}$, $[\![ P_i ]\!] = \langle (E_i, F_i, \mapsto_i, \rhd_i, \lambda_i, \mathsf{Act}_i), \mathsf{Init}_i, k_i \rangle$ and $\langle \mathcal{E}, \mathsf{Init}, k \rangle$ is composed of them as defined in the event structure semantics.
And there exist $\mathcal{E}'_Q$ and $\mathcal{E}'_R$ such that for $i \in \{Q, R\}$, $[\![ P_i ]\!] = \left\langle (E'_i, F'_i, \mapsto'_i, \rhd'_i, \lambda'_i, \mathsf{Act}'_i), \mathsf{Init}'_i, k'_i \right\rangle$ and $\left\langle \mathcal{E}', \mathsf{Init}', k' \right\rangle$ is composed of them as defined in the event structure semantics.
And by induction we have isomorphisms $f_Q : \mathcal{E}_Q \to \mathcal{E}'_Q$ and $f_R : \mathcal{E}_R \to \mathcal{E}'_R$ fulfilling the conditions.
We first define a helper function $f'(e) = \begin{cases} (f_R(e_R), f_Q(e_Q)) & \text{if } e = (e_Q, e_R) \\ (f_R(e_R), *) & \text{if } e = (*, e_R) \\ (*, f_Q(e_Q)) & \text{if } e = (e_Q, *) \end{cases}$, and then our isomorphism $f(e) = \begin{cases} (\{ f'(e'') \mid e'' \in X \}, f'(e')) & \text{if } e = (X, e') \\ f'(e) & \text{otherwise} \end{cases}$. Since the definition of parallel compositions treat both parts the same way, this clearly fulfils the conditions.

**$P = P_0 \mid (P_1 \mid P_2)$ and $P' = (P_0 \mid P_1) \mid P_2$:** Then there exist $\mathcal{E}_0$, $\mathcal{E}_1$, $\mathcal{E}_2$, and $\mathcal{E}_{1|2}$ such that $[\![ P_0 ]\!] = \langle \mathcal{E}_0, \mathsf{Init}_0, k_0 \rangle$, $[\![ P_1 ]\!] = \langle \mathcal{E}_1, \mathsf{Init}_1, k_1 \rangle$, $[\![ P_2 ]\!] = \langle \mathcal{E}_2, \mathsf{Init}_2, k_2 \rangle$, $\left\langle \mathcal{E}_{1|2}, \mathsf{Init}_{1|2}, k_{1|2} \right\rangle$ is made up of $\langle \mathcal{E}_1, \mathsf{Init}_1, k_1 \rangle$ and $\langle \mathcal{E}_2, \mathsf{Init}_2, k_2 \rangle$ as described in the parallel composition rule, and $\langle \mathcal{E}, \mathsf{Init}, k \rangle$ is made up of $\langle \mathcal{E}_0, \mathsf{Init}_0, k_0 \rangle$ and $\left\langle \mathcal{E}_{1|2}, \mathsf{Init}_{1|2}, k_{1|2} \right\rangle$ as described in the parallel composition rule.
And there exist $\mathcal{E}'_0$, $\mathcal{E}'_1$, $\mathcal{E}'_2$, and $\mathcal{E}_{0|1}$ such that $[\![ P_0 ]\!] = \left\langle \mathcal{E}'_0, \mathsf{Init}'_0, k'_0 \right\rangle$, $[\![ P_1 ]\!] = \left\langle \mathcal{E}'_1, \mathsf{Init}'_1, k'_1 \right\rangle$, $[\![ P_2 ]\!] = \left\langle \mathcal{E}'_2, \mathsf{Init}'_2, k'_2 \right\rangle$, $\left\langle \mathcal{E}_{0|1}, \mathsf{Init}_{0|1}, k_{0|1} \right\rangle$ is made up of $\langle \mathcal{E}_0, \mathsf{Init}_0, k_0 \rangle$ and $\langle \mathcal{E}_1, \mathsf{Init}_1, k_1 \rangle$ as described in the parallel composition rule, and $\left\langle \mathcal{E}', \mathsf{Init}', k' \right\rangle$ is made up of $\langle \mathcal{E}_Z, \mathsf{Init}_Z, k_Z \rangle$ and $\left\langle \mathcal{E}_{0|1}, \mathsf{Init}_{0|1}, k_{0|1} \right\rangle$ as described in the parallel composition rule. And there exist isomorphisms $f_0 : \mathcal{E}_0 \to \mathcal{E}'_0$, $f_1 : \mathcal{E}_1 \to \mathcal{E}'_1$, and $f_2 : \mathcal{E}_2 \to \mathcal{E}'_2$ satisfying the conditions of the proposition.

We define a helper function $f_{01}((e_0, e_1)) = (f_0(e_0), f_1(e_1))$ if $e_0 \in E_0$ and $e_1 \in E_1$ and define the morphism

$$f(e) = \begin{cases} (f_{01}((e_0, e_1)), f_2(e_2)) & \text{if } e = (e_0, (e_1, e_2)) \\ (Y, ((Y', f_{01}((e_0, e_1))), f_2(e_2))) & \text{if } e = (X, (e_0, (X', (e_1, e_2)))), \\ & Y' = \{f_{01}((e'_0, e'_1)) \mid \exists e'_2, X''.(e'_0, (X'', (e'_1, e'_2))) \in X \\ & \text{ and } e'_0 \in X_0 \in \mathsf{cause}(e_0) \text{ or } e'_1 \in X_1 \in \mathsf{cause}(e_1)\}, \\ & \text{and } Y = \{((f_{01}(Y''), f_{01}((e'_0, e'_1))), f_2(e'_2)) \in E_{0|1\times 2} \mid \\ & \exists X''.(e'_0, (X'', (e'_1, e'_2))) \in X \text{ and} \\ & \text{for all } (e''_0, e''_1) \in Y'', e'_0 \in X_0 \in \mathsf{cause}(e_0) \text{ or} \\ & \qquad e'_1 \in X_1 \in \mathsf{cause}(e_1) \text{ and} \\ & f_{01}(Y'') \cup \{f_{01}((e'_0, e'_1))\} \subseteq Y'\} \end{cases}$$

We first show that for any $e = (X, (e_0, (X', (e_1, e_2))))$, there exists at most one possible $f(e) \in E'$: Since causes must be conflict free, there can at most exist one $e'_2$ and $X''$ for each $e'_0$ and $e'_1$ such that $(e'_0, (X'', (e'_1, e'_2))) \in X$, meaning there can only exist one $Y'$ and $Y$ fulfilling the conditions.

We then show that for any $e = (X, (e_0, (X', (e_1, e_2))))$, there exists $f(e') = (Y, ((Y', (e'_0, e'_1)), e'_2)) \in E'$: By induction, $e'_0 \in E'_0$, $e'_1 \in E'_1$, and $e'_2 \in E'_2$, so we show that $(Y', (e'_0, e'_1)) \in E_{0|1}$. We know there exists $X_1 \in \mathsf{cause}(e_1)$ such that $X_1 \subseteq \pi_1(X') = \pi_1(\pi_{1|2}(X))$, and there exists $X_0 \in \mathsf{cause}(e_0)$ such that $X_0 \subseteq \pi_0(X)$. And since for all $e \in Y'$, either $e'_0 \in X_0 \in \mathsf{cause}(e_0)$ or $e'_Y \in X_Y \in \mathsf{cause}(e_1)$, we get that $Y' \in \mathsf{cause}((f_0(e_0), f_1(e_1)))$, and therefore $(Y', (f_0(e_0), f_1(e_1))) \in E_{0|1}$. And for similar reasons $Y \in \mathsf{cause}(((Y', (f_0(e_0), f_1(e_1))), f_2(e_2)))$, meaning $(Y, ((Y', (f_0(e_0), f_1(e_1))), f_2(e_2))) \in E'$.

We then show that for any $e' = (X, ((X', (e'_0, e'_1)), e'_2)) \in E'$, there exists $e = (Y, (e_0, (Y', (e_1, e_2)))) \in E$ such that $f(e) = e'$. By induction, there obviously exist $e_0, e_1, e_2$ such that $f_0(e_0) = e'_0$, $f_1(e_1) = e'_1$, and $f_2(e_2) = e'_2$. We also know there exist $X_0 \in \mathsf{cause}(e'_0)$, $X_1 \in \mathsf{cause}(e'_1)$, and $X_2 \in \mathsf{cause}(e'_2)$ such that (1) whenever $((X'', (e''_0, e''_1)), e''_2)) \in X$, either $e''_0 \in X_0$ or $e''_1 \in X_1$ or $e''_2 \in X_2$, and for each $(e'''_0, e'''_1) \in X''$, there exists $X''' \subseteq X''$ and $e'''_2$ such that $((X''', (e'''_0, e'''_1)), e'''_2) \in X$; and (2) whenever $e_i \in X_i$, there exists $((X'', (e''_0, e''_1)), e''_2)) \in X$ such that $e_i \in \{e''_0, e''_1, e''_2\}$.

For $i \in \{0, 1, 2\}$, since $f_i$ is an isomorphism, $f_i^{-1}(X_i) \in \mathsf{cause}(e_i)$, meaning if we set $Y' = \{(f_1^{-1}(e''_1), f_2^{-1}(e''_2)) \mid ((X'', (e''_0, e''_1)), e''_2)) \in X \text{ and } e''_1 \in X_1 \text{ or } e''_2 \in X_2\}$ and $Y = \{(f_0^{-1}(e''_0), (Y'', (f_1^{-1}(e''_1), f_2^{-1}(e''_2)))) \mid \exists X''.((X'', (e''_0, e''_1)), e''_2) \in X \text{ and } (Y'', (f_1^{-1}(e''_1), f_2^{-1}(e''_2))) \in Y'\}$, we have $e = (Y, (e_0, (Y', (e_1, e_2)))) \in E$ and $f(e) = e'$.

We then show that $f$ is a morphism, meaning for $e, e' \in E$:

- Obviously $\lambda(e) = \lambda'(f(e))$.
- If $f(e) = f(e')$ then either $e = (e_0, (e_1, e_2)) = e'$, or $e = (X, (e_0, (Y, (e_1, e_2))))$ and $e = (X', (e_0, (Y', (e_1, e_2))))$, and $(e''_0, (Y'', (e''_1, e''_2))) \in X$ if and only if there exists $(e''_0, (Y''', (e''_1, e''_2))) \in X'$. However, since $Y'', Y''' \in \mathsf{cause}(e''_1, e''_2)$,

either $Y'' = Y'''$, or there exist $y'' \in Y''$ and $y''' \in Y'''$ such that $y'' \sharp 1 \mid 2 y'''$. And in addition, there exist $e_0'''', e_0'''', Y_{y''}$, and $Y_{y'''}$ such that $(e_0'''', (Y_{y''}, y'')) \in X$ and $(e_0'''', (Y_{y'''}, y''')) \in X'$. Since $X$ and $x'$ must be conflict-free, $X = X'$.

- If $X \mapsto' f(e)^*$, then either $e^* = (e_0, (e_1, e_2))$, $e^* = (Y, (e_0, (Y', (e_1, e_2))))$, $e^* = \underline{(e_0, (e_1, e_2))}$, or $e^* = \underline{(Y, (e_0, (Y', (e_1, e_2))))}$.
  If $e = \underline{(e_0, (e_1, e_2))}$ then there exists $i \in 0, 1 \mid 2$ and $X_i$ such that $X_i \mapsto_i \pi_i(e)$, and $X = \{e'' \mid \pi_i(e'') \in X_i\}$, meaning if $i = 0$, then $\{e'' \mid \pi_0(e'') \in X_i\} \mapsto_{01} (e_0, e_1)$ and therefore $\{e'' \mid \pi_0(\pi_{0|1}(e'')) \in X_i\} \mapsto ((e_0, e_1), e_2)$, and obviously $f(\{e'' \mid \pi_0(\pi_{0|1}(e'')) \in X_i\}) = X$. If $i = 1 \mid 2$ then there exists $j \in 1, 2$ and $X_j$ such that $X_j \mapsto_j \pi_j(e)$, and $X = \{e'' \mid \pi_j(\pi_{1|2}(e'')) \in X_j\}$, and by similar logic if $j = 1$ then $\{e'' \mid \pi_1(\pi_{0|1}(e'')) \in X_j\} \mapsto ((e_0, e_1), e_2)$, and $f(\{e'' \mid \pi_1(\pi_{0|1}(e'')) \in X_j\}) = X$ and if $j = 2$ then $\{e'' \mid \pi_2(e'') \in X_j\} \mapsto ((e_0, e_1), e_2)$, and $f(\{e'' \mid \pi_2(e'') \in X_j\}) = X$.
  If $e = (Y, (e_0, (Y', (e_1, e_2))))$ and $f(e) = (Z, ((Z', (f_0(e_0), f_1(e_1))), f_2(e_2)))$ then there exists $e' = ((Z'', (e_0', e_1')), e_2') \in Z$ such that $X = \{(X', e') \mid X' \subseteq Z\}$. This means there exists $Y'', e_0'', e_1'', e_2''$ such that $f_0(e_0'') = e_0'$, $f_1(e_1'') = e_1'$, $f_2(e_2'') = e_2'$, and $(e_0'', (Y'', (e_1'', e_2''))) \in Y$. Additionally, either $(Z'', (e_0', e_1')) \in \mathsf{cause}((Z', (f_0(e_0), f_1(e_1)))$ or $e_2' \in \mathsf{cause}(e_2)$. And if $(Z'', (e_0', e_1')) \in \mathsf{cause}((Z', (f_0(e_0), f_1(e_1)))$ then either $\{e_0''' \mid (e_0''', e_1''') \in Z'\} \in \mathsf{cause}(f_0(e_0))$ or $\{e_1''' \mid (e_0''', e_1''') \in Z'\} \in \mathsf{cause}(f_1(e_1))$ We therefore get $\{(Y''', (e_0'', (Y'', (e_1'', e_2'')))) \mid Y''' \subseteq Y\} \mapsto e$, and $f(\{(Y''', (e_0'', (Y'', (e_1'', e_2'')))) \mid Y''' \subseteq Y\}) = X$.
  If $e^* = \underline{(e_0, (e_1, e_2))}$, or $e^* = \underline{(Y, (e_0, (Y', (e_1, e_2))))}$, the cases are similar to the previous two.

We can use similar logic to argue that $f^{-1}$ is a morphism.

**$P = P' \mid 0$:** Then there exists $\mathcal{E}$ and $\mathcal{E}'$ such that $\{\!|P'|\!\} = \left\langle \mathcal{E}', \mathsf{Init}', k' \right\rangle$, $\{\!|P|\!\} = \langle \mathcal{E}, \mathsf{Init}, k \rangle$, and $\mathcal{E}$ is composed of $\mathcal{E}'$ and the empty LREBES, $\mathcal{E}_0$ as described in the parallel composition rule.

We define $f(e) = \begin{cases} e' & \text{if } e = (X, (e', *)) \\ e' & \text{if } e = (e', *) \end{cases}$

And show that $f : \mathcal{E} \to \mathcal{E}'$ is a morphism, meaning for all $e_0, e_1 \in E$:

- Clearly $\lambda(e_0) = \lambda'(f(e_0))$.
- If $f(e_0) = f(e_1)$ then there exists $e$ such that either $e_0 = (e, *) = e_1$ or there exist $X_0$ and $X_1$ such that $e_0 = (X_0(e, *))$ and $e_1 = (X_1(e, *))$. However, by Lemma E.7, we know that whenever $X_0' \mapsto e$, $X_0'$ contains exactly one event, $e_0'$. Since that event cannot synchronise with anything from $E_0$, $e_0'$ must be in every possible cause of $e$, and similarly for the causes of $e_0''$, meaning $e$ can only have one cause in $\mathcal{E}' || \mathcal{E}_0$, and therefore $e_0 = e_1$.
- For $X' \subseteq E'$, if $X' \mapsto' f(e_0)^*$ then $f(e_0) = e_0'$ and either $e_0 = (e_0', *)$ or $e_0 = (X_0, (e_0', *))$.
  If $e_0^* = (e_0', *)$, then $\{e \mid \exists e' \in X'. e = (X, (e', *)) \text{ or } e = (e', *)\} \mapsto e_0$. Clearly $\{e \mid \exists e' \in X'. e = (X, (e', *)) \text{ or } e = (e', *)\} = \{e \mid f(e) \in X'\}$
  If $e_0^* = (X_0, (e_0', *))$ then by Lemma E.7 there exists $e$ such that $X' = \{e\}$. Clearly this requires that $(e, *) \in X_0$, which means $\{(X_0', (e, *)) \mid X_0' \subseteq X_0\} \mapsto e_0$, and clearly $f(\{(X_0', (e, *)) \mid X_0' \subseteq X_0\}) = \{e\}$.

If $e_0^* = (e_0', *)$ then $\{e \mid e = (X, (e', *)) \text{ or } e = (', *) \text{ for } e' \in X'\} \mapsto e_0^*$.

If $e_0^* = \overline{(X_0, (e_0', *))}$ then $\bigcup\{X'' \mid \exists X''' \in E'.X''' \mapsto' \underline{e_0'} \text{ or } \exists (e', *) \in X_0.X'' \mapsto' \underline{e'}, \text{ and } e'' \in X'' \text{ iff } f(e'') \in X'''\} \mapsto e_0^*$, by Lemmas E.9 and E.10, we know that for all $e \in X_0$, if $X'' \mapsto \underline{e}$, then $X'' \subseteq X'$, meaning $X' = \bigcup\{X'' \mid \exists X''' \in E'.X''' \mapsto' \underline{e_0'} \text{ or } \exists (e', *) \in X_0.X'' \mapsto' \underline{e'}, \text{ and } e'' \in X'' \text{ iff } f(e'') \in X'''\}$.

– If $f(e_0) \rhd f(e_1)^*$ then by definition, $e_0 \rhd e_1\ *$.

We then prove $f$ is bijective: We already showed above, that $f$ is injective, and it is clear that it is also surjective.

In order to show $f$ is an isomorphism, we therefore only need to show that $f^{-1}$ is a morphism, meaning for $e_0', e_1' \in E'$:

– Again, clearly $\lambda(f^{-1}(e_0')) = \lambda'(e_0')$.
– If $f^{-1}(e_0') = f^{-1}(e_1')$ then we already know $f$ is a bijection, so $e_0' = e_1'$.
– For $X \subseteq E$, if $X \mapsto f^{-1}(e_0')^*$ then $f^{-1}(e_0') = e_0$ and either $e_0 = (e_0', *)$ or $e_0 = (X_0, (e_0', *))$.

  If $e_0^* = (e_0', *)$, then $\{e \mid (e, *) \in X \text{ or } \exists X'.(X', (e, *)) \in X\} \mapsto e_0'$.

  If $e_0^* = (X_0, (e_0', *))$ then by Lemma E.7 we know there exists an $e$ such that $X = \{e\}$. This means there exists $X'$ such that $e = (X', (e', *))$ and $(e', *) \in X_0$, meaning $\{e'\} \mapsto' e_0'$.

  If $e_0^* = (e_0', *)$ then either $X = \{e_0\}$, and obviously $\{e_0'\} \mapsto \underline{e_0'}$, or there exists an $X'$ such that $\overline{X' \mapsto e_0}$ and $X = \{e \mid \exists e' \in X'.e = (e', *) \text{ or } e = (X'', (e', *))\}$.

  If $e_0^* = \overline{(X_0, (e_0', *))}$ then either $X = \{e_0\}$, and obviously $\{e_0'\} \mapsto \underline{e_0'}$, or $X = \bigcup\{X'' \mid \overline{f(X'') \mapsto'} \underline{e_0'} \text{ or } \exists (e', *) \in X_0.f(X'') \mapsto' \underline{e'}\}$. Clearly any of these $X''$s can be used to fulfil the condition.
– If $f^{-1}(e_0') \rhd f^{-1}(e_1')^*$ then either (1) $e_0' \rhd e_1'^*$, (2) $e_0' = e_1'$ and $f^{-1}(e_0') \neq f^{-1}(e_1')^*$, (3) $e_1'^* = e_1'$, $f^{-1}(e_0') \neq f^{-1}(e_1')^*$, and $f^{-1}(e_0') \in X \mapsto \overline{f^{-1}(e_1')}$, or (4) $e_1'^* = e_1'$ and there exist $\gamma_0$ and $\gamma_1$ such that $\lambda(f^{-1}(e_0')) \in \{\mathsf{roll}\ \gamma_0, \mathsf{roll\ bound}\}$ and $\lambda(f^{-1}(e_1)) \in \{\mathsf{roll}\ \gamma_1, \mathsf{roll\ bound}\}$.

  In case 1, the condition is trivially fulfilled. Case 2 will never occur. In case 3, as shown above, $e_0' \in f(X) \mapsto \underline{e_1'}$, and by Lemma E.8, this means $e_0' \rhd e_1'$. In case 4, since the $e_0'$ and $e_1'$ must both have been caused by a roll at the end of a subprocess, they were either in parallel or different option in a choice, and in either case clearly $e_0' \rhd' e_1'$.

And obviously from Lemma E.6 and the definition of $\mathsf{Init}$, we see that $f(\mathsf{Init}) = \mathsf{Init}'$ and $f \circ k' = k$.

$P = X + Y$ **and** $P' = Y + X$**:** Selection works the same in roll-CCSK as in CCSK, so this case is the same as in Proposition 4.9.

$P = (X + Y) + Z$ **and** $P' = (X + Y) + Z$**:** Selection works the same in roll-CCSK as in CCSK, so this case is the same as in Proposition 4.9.

$P = P' + 0$**:** Selection works the same in roll-CCSK as in CCSK, so this case is the same as in Proposition 4.9.

$P = Q \setminus A$, $P' = Q' \setminus A$, **and** $Q \equiv Q'$**:** Then $\{\!|Q|\!\} = \left\langle \mathcal{E}_Q, \mathsf{Init}_Q, k_Q \right\rangle$, and $\{\!|Q'|\!\} = \left\langle \mathcal{E}_{Q'}, \mathsf{Init}_{Q'}, k_{Q'} \right\rangle$, there exist an isomorphism $f_Q : \mathcal{E}_Q \to \mathcal{E}_{Q'}$ such that $f_Q(\mathsf{Init}_Q) = \mathsf{Init}_{Q'}$ and for all $e \in \mathsf{Init}_Q$, $k_Q(e) = k_{Q'}(f_Q(e))$, and
$\langle \mathcal{E}, \mathsf{Init}, k \rangle = \left\langle \mathcal{E}_Q \upharpoonright \rho(A \cup \overline{A}), \mathsf{Init}_Q \cap \rho(A \cup \overline{A}), k_Q \upharpoonright \rho(A \cup \overline{A}) \right\rangle$ and $\left\langle \mathcal{E}', \mathsf{Init}', k' \right\rangle = \left\langle \mathcal{E}_{Q'} \upharpoonright \rho(A \cup \overline{A}), \mathsf{Init}_{Q'} \cap \rho(A \cup \overline{A}), k_{Q'} \upharpoonright \rho(A \cup \overline{A}) \right\rangle$.
We now show that $e \in \rho(A \cup \overline{A})$ if and only if $f(e) \in \rho(A \cup \overline{A})$.
For any $e \in E_Q$, obviously $\lambda_Q(e) \in A \cup \overline{A}$ iff $\lambda_{Q'}(f(e)) \in A \cup \overline{A}$. We show that for any $X \subseteq E_Q$, $X \in \mathsf{causes}(e)$ if and only if $f(X) \in \mathsf{cause}(f(e))$ by induction in the size of $X$.
If $X = \emptyset$ then there does not exist $x \subseteq E_Q$ such that $x \mapsto_Q e$, and by definition of an morphism, there cannot exist $x' \subseteq E_{Q'}$ such that $x' \mapsto f(e)$, meaning $\emptyset \in \mathsf{cause}(e)$. And since $f$ is an isomorphism the same argument can be used for $f^{-1}$.
If $X$ contains $n$ events, and for all events $e'$ and $X' \in \mathsf{cause}(e')$ such that $X'$ contains less that $n$ events, $X' \subseteq \rho(A \cup \overline{A})$ if and only if $f(X') \in \mathsf{cause}(f(e'))$ then whenever $x' \mapsto_{Q'} f(e)$, there exists $x \subseteq E_Q$ such that $x \mapsto_Q e$ and $f(x) \subseteq x'$, meaning there exists $e''$ such that $x \cap X = \{e''\}$, and $x' \cap f(X) \supseteq \{f(e'')\}$. And by induction if $X'' \mapsto e'' \in X$ then $X'' \subset X$ and therefore $f(X'') \in \mathsf{causes}(e'')$. And since $X$ is conflict-free, obviously $f(X)$ is conflict free. And since $f$ is an isomorphism the same argument can be used for $f^{-1}$.

$P = A\left\langle \tilde{b}, \tilde{\delta} \right\rangle$ **and** $P' = (\nu\ \tilde{\delta}) P_A\{^{\tilde{b},\tilde{\delta}}/_{\tilde{a},\tilde{\gamma}}\}$ **where** $A\langle \tilde{a}, \tilde{\gamma} \rangle = P_A$**:** Obvious

$P \equiv_\alpha P'$ Obvious.

All the structural induction on bound tags cases follow naturally from induction.

### E.9 Proof of Theorem 7.5

*Proof.* We first prove that if there exists a $P'$ and transition $P \xrightarrow{\mu_\gamma[m]} P'$ then there exists an isomorphism $f : \mathcal{E} \to \mathcal{E}'$ and a transition $\mathsf{Init} \xrightarrow{\{e\}} X$ such that $\lambda(e) = \mu$, $f \circ k' = k[e \mapsto m]$, and $f(X) = \mathsf{Init}'$. We prove this by induction on the transition $P \xrightarrow{\mu[m]} P'$:

- Suppose $P = \alpha_\gamma.Q$, $P' = \alpha_\gamma[m].Q$, $\mu = \alpha$, and $\mathsf{std}(Q)$. Then there exist $\mathcal{E}_Q$ and $e_\alpha$ such that $\{\!|Q|\!\} = \left\langle (E_Q, F_Q, \mapsto_Q, \rhd_Q, \lambda_Q, \mathsf{Act}_Q), \mathsf{Init}, k \right\rangle$ and $\langle \mathcal{E}, \mathsf{Init}, k \rangle$ is constructed based on this as described in the prefix rule.
  And there exist $\mathcal{E}_{Q'}$ and $e'_\alpha$ such that $\{\!|Q|\!\}_l = \left\langle (E_{Q'}, F_{Q'}, \mapsto_{Q'}, \rhd_{Q'}, \lambda_{Q'}, \mathsf{Act}_{Q'}), \mathsf{Init}_{Q'}, k_{Q'} \right\rangle$ and $\left\langle \mathcal{E}', \mathsf{Init}', k' \right\rangle$ is constructed from this a described in the past prefix rule.
  By induction, there must exist an isomorphism $f_Q : \mathcal{E}_Q \to \mathcal{E}_{Q'}$, and we define $f = f_Q[e_\alpha \mapsto e'_\alpha]$, which is clearly an isomorphism.
  Since $\mathsf{std}(Q)$, ,meaning $\mathsf{Init} = \emptyset$, and since no $X$ exists such that $X \mapsto e_\alpha$, $\mathsf{Init} \xrightarrow{e_\alpha} \{e_\alpha\}$, and the rest of the conditions are obviously satisfied.
- Suppose that $P = \alpha_\gamma[n].Q$, $P' = \alpha_\gamma[n].Q'$, $Q \xrightarrow{\mu[m]} Q'$, and $n \neq m$.
  Then there exist $\mathcal{E}_Q$ and $e_\alpha$ such that $\{\!|Q|\!\} = \left\langle (E_Q, F_Q, \mapsto_Q, \rhd_Q, \lambda_Q, \mathsf{Act}_Q), \mathsf{Init}_Q, k_Q \right\rangle_l$
  Then there exist $\mathcal{E}_Q$ and $e_\alpha$ such that $\{\!|Q|\!\} = \left\langle (E_Q, F_Q, \mapsto_Q, \rhd_Q, \lambda_Q, \mathsf{Act}_Q), \mathsf{Init}_Q, k_Q \right\rangle$
  and $\langle \mathcal{E}, \mathsf{Init}, k \rangle$ is constructed based on this as described in the past prefix rule.

And there exist $\mathcal{E}_{Q'}$ and $e'_\alpha$ such that $[\![Q]\!]_l = \left\langle (E_{Q'}, F_{Q'}, \mapsto_{Q'}, \rhd_{Q'}, \lambda_{Q'}, \mathsf{Act}_{Q'}), \mathsf{Init}_{Q'}, k_{Q'} \right\rangle$ and $\left\langle \mathcal{E}', \mathsf{Init}', k' \right\rangle$ is constructed from this a described in the past prefix rule.

By induction, we get an isomorphism $f_Q : \mathcal{E}_Q \to \mathcal{E}_{Q'}$ and a transition $\mathsf{Init}_Q \xrightarrow{\{e\}} X_Q$ in $C_{er}(\mathcal{E}_Q)$ such that $\lambda_Q(e) = \mu$, $k_{Q'}(f_Q(e)) = m$, and $f_Q(X_Q) = \mathsf{Init}_{Q'}$.
We define $f = f_Q[e_\alpha \mapsto e'_\alpha]$. Since $\mathsf{Init}_Q$ and $X_Q$ are conflict-free in $\mathcal{E}_Q$, $\mathsf{Init}_Q \cup \{e_\alpha\} = \mathsf{Init}$ and $X_Q \cup \{e_\alpha\} = X$ are configurations of $C_{er}(\mathcal{E})$, and clearly $\mathsf{Init} \xrightarrow{\{e\}} X$.

– Suppose $P = P_0 \mid P_1$, $P' = P'_0 \mid P_1$, $P_0 \xrightarrow{\mu[m]} P'_0$, and $\mathsf{fsh}[m](P_1)$. Then there exist $\mathcal{E}_0$ and $\mathcal{E}_1$ such that for $i \in \{0, 1\}$, $[\![P_i]\!] = \langle (E_i, F_i, \mapsto_i, \rhd_i, \lambda_i, \mathsf{Act}_i), \mathsf{Init}_i, k_i \rangle$, and $\langle \mathcal{E}, \mathsf{Init}, k \rangle$ is constructed as described in the parallel composition rule.
And there exist $\mathcal{E}_{0'}$ and $\mathcal{E}_{1'}$ such that $[\![P_{0'}]\!] = \langle (E_{0'}, F_{0'}, \mapsto_{0'}, \rhd_{0'}, \lambda_{0'}, \mathsf{Act}_{0'}), \mathsf{Init}_{0'}, k_{0'} \rangle$, $[\![P_1]\!] = \langle (E_{1'}, F_{1'}, \mapsto_{1'}, \rhd_{1'}, \lambda_{1'}, \mathsf{Act}_{1'}), \mathsf{Init}_{1'}, k_{1'} \rangle$, and $\left\langle \mathcal{E}', \mathsf{Init}', k' \right\rangle$ is constructed as described in the parallel composition rule.
We have isomorphisms $f_0 : \mathcal{E}_0 \to \mathcal{E}_{0'}$ and $f_1 : \mathcal{E}_1 \to \mathcal{E}_{1'}$, and there exists a transition $\mathsf{Init}_0 \xrightarrow{e_\mu} X$ in $C_{er}(\mathcal{E}_0)$ such that $\lambda_0(e_\mu) = \mu$, $k_{0'}(f_0(e)) = m$ and $f_0(X) = \mathsf{Init}_{0'}$.

We define a helper function $f'(e) = \begin{cases} (f_0(e_0), *) & \text{if } e = (e_0, *) \\ (*, f_1(e_1)) & \text{if } e = (*, e_1) \\ (f_0(e_0), f_1(e_1)) & \text{if } e = (e_0, e_1) \end{cases}$ and our isomorphism as: $f(e) = \begin{cases} (f'(X), f'(e')) & \text{if } e = (X, e') \\ f'(e) & \text{otherwise} \end{cases}$.

It is clear that $f \circ f^{-1} = I_{\mathcal{E}}$ and $f^{-1} \circ f = I_{\mathcal{E}'}$
We show that $f : \mathcal{E} \to \mathcal{E}'$ is a morphism, meaning for all $e, e' \in E$:

- Obviously $\lambda(e) = \lambda'(f(e))$
- If $f(e) = f(e')$ then since $f_0$ and $f_1$ are injective, $e = e'$.
- For $X' \subseteq E'$, if $X' \mapsto' f(e)^*$ then either $e^* = (Y, e_\times$ and there exists $e_{\times'} \in f(Y)$ such that $X' = \{(Y', e_{\times'}) \mid Y' \subseteq f'(Y)\}$, or $e^* \in E_\times$ and there exists $X''$ such that $X'' \mapsto'_\times f(e)$ and $X' = \{e' \mid (\pi'_0(e'), \pi'_1(e')) \in X''\}$, or $e^* = \underline{(Y, e_\times)}$ and $X' = \{f(e)\}$, or $e^* = \underline{(Y, e_\times)}$ and

$$X' = \bigcup \left\{ X'' \;\middle|\; \begin{array}{l} \exists i \in \{0,1\}, X_i \in E'_i . X'_i \mapsto'_i \underline{\pi'_i(f(e))} \\ \text{ or } \exists e_{\times'} \in f(Y) . X'_i \mapsto'_i \underline{\pi'_i(e_{\times'})} \\ \text{, and } e'' \in X'' \text{ iff } \underline{\pi'_i(e'') \in X_i} \end{array} \right\}$$

  or $e^* \in \underline{E_\times}$ and there exists $X''$ such that $X'' \mapsto'_\times \underline{f(e)}$ and $X' = \{e' \mid (\pi'_0(e'), \underline{\pi'_1(e')}) \in X''\}$.
  If $e^* = (Y, e_\times$ and there exists $e_{\times'} \in f(Y)$ such that $X' = \{(Y', e_{\times'}) \mid Y' \subseteq f(Y)\}$ then there exists an $e'_{times} \in Y$ such that $f(e'_\times) = e_{\times'}$ and clearly $\{(Y', e'_\times) \mid Y' \subseteq Y\} \mapsto (Y, e_\times)$.
  If $e^* \in E_\times$ and there exists $X''$ such that $X'' \mapsto'_\times f(e)$ and $X' = \{e' \mid (\pi'_0(e'), \pi'_1(e')) \in X''\}$ then by induction and since $||$ is an REBES product, there exists $X''' \subseteq E_\times$ such that $X''' \mapsto_\times e$, $f(X''') \subseteq X''$, and if $e' \in X'''$ then $f(e') \neq \bot$. This means $\{e' \mid (\pi_0(e'), \pi_1(e')) \in X'''\} \mapsto e$.

If $e^* = \overline{(Y, e_\times)}$ and $X' = \{f(e)\}$, or $e^* = \overline{(Y, e_\times)}$ and $X' = \bigcup \left\{ X'' \;\middle|\; \begin{array}{l} \exists i \in \{0,1\}, X'_i \in E'_i . X'_i \mapsto'_i \overline{\pi'_i(f(e))} \\ \text{ or } \exists e_{\times'} \in f(Y) . X'_i \mapsto'_i \overline{\pi'_i(e_{\times'})} \\ \text{, and } e'' \in X'' \text{ iff } \overline{\pi'_i(e'') \in X_i} \end{array} \right\}$ then by induction, since $f^{-1}$ is a morphism for each $X_i \mapsto_i \overline{\pi_i(e)}$, $f(X_i) \subset X'_i \mapsto'_i \overline{\pi'_i(f(e))}$, and for each $X_i \mapsto_i \overline{\pi_i(e'_\times)} \in \underline{Y}$, $f(X_i) \subset X'_i \mapsto'_i \overline{\pi'_i(f(e'_\times))}$ meaning $\left\{ X'' \;\middle|\; \begin{array}{l} \exists i \in \{0,1\}, X_i \in E_i . X_i \mapsto_i \overline{\pi_i(e)} \\ \text{ or } \exists e'_\times \in Y . X_i \mapsto_i \pi_i(e'_\times) \\ \text{, and } e'' \in X'' \text{ iff } \overline{\pi'_i(e'') \in X_i} \end{array} \right\} \subseteq \left\{ X'' \;\middle|\; \begin{array}{l} \exists i \in \{0,1\}, X_i \in E'_i . X'_i \mapsto'_i \overline{\pi'_i(f(e))} \\ \text{ or } \exists e_{\times'} \in f(Y) . X'_i \mapsto'_i \overline{\pi'_i(e_{\times'})} \\ \text{, and } e'' \in X'' \text{ iff } \overline{\pi'_i(e'') \in X_i} \end{array} \right\}$.
If $e^* \in \underline{E_\times}$ and there exists $X''$ such that $X'' \mapsto'_\times \underline{f(e)}$ and $X' = \{e' \mid (\pi'_0(e'), \overline{\pi'_1(e')}) \in X''\}$ then by induction and because $||$ is an REBES product, there exists $X''' \subseteq E_\times$ such that $f(X''') \subseteq X''$ and $X''' \mapsto_\times \underline{e}$. This means $\{e' \mid (\pi_0(e'), \pi_1(e')) \in X'''\} \mapsto \underline{e}$.

- If $f(e) \rhd' f(e')^*$ then there exists $i \in \{0,1\}$ such that either $\pi'_i(f(e)) \rhd'_i \pi'_i(f(e'))^*$, or $\pi'_i(f(e)) = \pi'_i(f(e')) \neq \bot$, and $f(e) \neq f(e')$, or $f(e')^* = f(e')$, $f(e) \neq f(e')$, and $f(e) \in X \mapsto' \underline{f(e')}$, or $f(e')^* = f(e')$ and there exist $\gamma, \gamma'$ such that $\lambda'(f(e)) \in \{\mathsf{roll}\ \gamma, \mathsf{roll\ bound}\}$ and $\lambda'(f(e')) \in \{\mathsf{roll}\ \gamma', \mathsf{roll\ bound}\}$.
  If $\pi'_i(f(e)) \rhd'_i \pi'_i(f(e'))^*$ then by induction $\pi_i(e) \rhd_i \pi_i(e')^*$, meaning $e^* \rhd e$.
  If $\pi'_i(f(e)) = \pi'_i(f(e')) \neq \bot$, and $f(e) \neq f(e')$ then $\pi_i(e) = \pi_i(e') \neq \bot$ and $e \neq e'$, meaning $e \rhd e'^*$.
  If $f(e')^* = f(e')$, $f(e) \neq f(e')$, and $f(e) \in X \mapsto' \underline{f(e')}$ then, since, by similar arguments to the previous case, $f^{-1}(X) \mapsto \underline{f(e')}$, and $e \in f^{-1}(X)$, $e \rhd e^*$.
  If $f(e')^* = f(e')$ and there exist $\gamma, \gamma'$ such that $\lambda'(f(e)) \in \{\mathsf{roll}\ \gamma, \mathsf{roll\ bound}\}$ and $\lambda'(f(e')) \in \{\mathsf{roll}\ \gamma', \mathsf{roll\ bound}\}$, then $\lambda(e) \in \{\mathsf{roll}\ \gamma, \mathsf{roll\ bound}\}$ and $\lambda(e') \in \{\mathsf{roll}\ \gamma', \mathsf{roll\ bound}\}$, meaning $e \rhd e'^*$.

By similar arguments, $f^{-1}$ is a morphism too.

We now show that there exists an event $(Y, (e_\mu, *)) \in E$ such that $\{e' \mid (\pi_0(e'), \pi_1(e')) \in Y\} \subseteq \mathsf{Init}$. Since $\mathsf{Init}_0 \xrightarrow{\{e_\mu\}}$, for every $X_0 \mapsto_0 e_\mu$, $X_0 \cap \mathsf{Init}_0 = X_0 = \{e_0\}$, and if $X'_0 \mapsto_0 e_0$ then by Lemma E.6, $X'_0 \mapsto_0 e_\mu$, and therefore $X'_0 \cap \mathsf{Init}_0 \neq \emptyset$. Therefore there must exist one $(Y, (e_\mu, *)) \in E$ such that $\{e' \mid (\pi_0(e'), \pi_1(e')) \in Y\} \subseteq \mathsf{Init}$.

We use this $(Y, (e_\mu, *))$ as our $e$ and show that $\mathsf{Init} \xrightarrow{\{e\}}$: Since $Y \subseteq \mathsf{Init}$, for every $X \mapsto e$, $X \cap \mathsf{Init} \neq \emptyset$. And if $e' \rhd e$ then it must that either $\pi_0(e') \rhd_0 e_\mu$, in which case $\pi_0(e') \notin \mathsf{Init}_0$, and therefore $e' \notin \mathsf{Init}$, or $\pi_0(e') = e_\mu$ and $e \neq e'$, in which case, since $\mathsf{Init}_0 \xrightarrow{\{e_\mu\}}$, $e' \notin \mathsf{Init}_0$, and therefore $e' \notin \mathsf{Init}$, or $e' \in X \mapsto \underline{e}$ and $e' \neq e$, in which case $\pi_0(e') \in X_0 \mapsto e_\mu$ or $\pi_0(e') \in X_0 \mapsto \pi_0(e'')$ for $e'' \in Y$, and by Lemmas E.8 and E.10, $\pi_0(e') \rhd e_\mu$, meaning $\pi_0(e') \notin \mathsf{Init}_0$, and $e' \notin I$.

We therefore have $\mathsf{Init} \xrightarrow{\{e\}} I \cup \{e\}$, and obviously $\lambda(e) = \lambda_0(e_\mu) = \mu$ and $f \circ k' = k[e \mapsto m]$, and since $f_0(\mathsf{Init}_0 \cup \{e_\mu\}) = \mathsf{Init}'_0$ and $f_1(\mathsf{Init}_1) = \mathsf{Init}'_1$, and there only exists one $(Y, (e_\mu, *)) \in E$ such that $\{e' \mid (\pi_0(e'), \pi_1(e')) \in Y\} \subseteq \mathsf{Init}$, $f(\mathsf{Init} \cup \{e\}) = \mathsf{Init}'$.

- Suppose $P = P_0 \mid P_1$, $P' = P'_0 \mid P'_1$, $P_0 \xrightarrow{\alpha[m]} P'_0$, $P_1 \xrightarrow{\overline{\alpha}[m]} P'_1$, and $\mu = \tau$.
  Then the construction of $\langle \mathcal{E}, \mathsf{Init}, k\rangle$ and $\left\langle \mathcal{E}', \mathsf{Init}', k' \right\rangle$ and the isomorphisms is similar to the previous case. And by induction we have transitions $\mathsf{Init}_0 \xrightarrow{\{e_0\}}$ and $\mathsf{Init}_1 \xrightarrow{\{e_1\}}$ fulfilling the conditions.
  For similar reasons to the previous case there exists exactly one $(Y, (e_0, e_1)$ such that $\{e' \mid (\pi_0(e'), \pi_1(e')) \in Y\} \subseteq \mathsf{Init}$, and we use this $(Y, (e_0, e_1))$ as $e$, and the rest of the proof follows similarly.
- Suppose $P = P_0 + P_1$, $P' = P'_0 + P_1$, $P_0 \xrightarrow{\mu[m]} P'_0$, and $\mathsf{std}(P_1)$. Then the rule for selection is the same in roll-CCSK as in CCSK, and the case is therefore identical to Theorem 4.10.
- Suppose $P = Q \setminus A$, $P' = Q' \setminus A$, $Q \xrightarrow{\mu[m]} Q'$, and $\mu \notin A \cup \overline{A}$. Then there exist $\mathcal{E}_Q$ and $\mathcal{E}'_Q$ such that $\{\!|Q|\!\} = \left\langle \mathcal{E}_Q, \mathsf{Init}_Q, k\right\rangle$, $\{\!|Q'|\!\} = \left\langle \mathcal{E}'_Q, \mathsf{Init}'_Q, k'_Q \right\rangle$, and $\mathcal{E}$ and $\mathcal{E}'$ are constructed from $\mathcal{E}_Q$ and $\mathcal{E}'_Q$ as described in the restriction rule, and there exists an isomorphism $f_Q : \mathcal{E}_Q \to \mathcal{E}'_Q$ and a transition $\mathsf{Init}_Q \xrightarrow{\{e_Q\}}$ where $\lambda_Q(e_Q) = \mu$, $f_Q \circ k'_Q = k_Q[e_q \mapsto m]$, and $f_Q(\mathsf{Init}_Q \cup \{e_Q\}) = \mathsf{Init}'_Q$.
  Since there exists a standard process $P''$ such that $P'' \to^* P$, there cannot exist $e' \in \mathsf{Init}$ such that $\lambda(e') \in A \cup \overline{A}$ or for all $x \in \mathsf{cause}(e')$, there exists $e'' \in x$ such that $\lambda(e'') \in A \cup \overline{A}$, meaning $\mathsf{Init} \cap \rho(A \cup \overline{A}) = \mathsf{Init}$, and, since $e_Q \in \rho(A \cup \overline{A})$, $\mathsf{Init} \xrightarrow{e_Q}$.
- Suppose $P = Q[f]$, $P' = Q'[f]$, $Q \xrightarrow{\nu[m]} Q'$, and $f'(\nu) = \mu$. Then the rule for functions is the same in roll-CCSK as in CCSK, and the case is therefore identical to Theorem 4.10.
- Suppose $P = (\nu\ \gamma)Q$, $P' = (\nu\ \gamma)Q'$, and $Q \xrightarrow{\mu[m]} Q'$. Then there exist $\lambda_Q$ and $\mathsf{Act}_Q$ such that $\{\!|Q|\!\} = \left\langle (E, F, \mapsto, \triangleright, \lambda_Q, \mathsf{Act}_Q), \mathsf{Init}, k\right\rangle$ and $\langle \mathcal{E}, \mathsf{Init}, k\rangle$ is constructed based on this as described in the tag binding rule. And there exist $\lambda_{Q'}$ and $\mathsf{Act}_{Q'}$ such that $\{\!|Q'|\!\} = \left\langle (E', F', \mapsto', \triangleright', \lambda_{Q'}, \mathsf{Act}_{Q'}), \mathsf{Init}', k' \right\rangle$ and $\left\langle \mathcal{E}', \mathsf{Init}', k' \right\rangle$ is constructed based on this as described in the tag binding rule.
  And there exists an isomorphisms $f_Q : (E, F, \mapsto, \triangleright, \lambda_Q, \mathsf{Act}_Q) \to (E', F', \mapsto', \triangleright', \lambda_{Q'}, \mathsf{Act}_{Q'})$ and a transition $\mathsf{Init} \xrightarrow{e} X$ in $C_{er}((E, F, \mapsto, \triangleright, \lambda_Q, \mathsf{Act}_Q))$ such that $\lambda_Q(e) = \mu$, $f \circ k' = k[e \mapsto m]$, and $f(X) = \mathsf{Init}'$. Clearly this all still holds for $\mathcal{E}$ and $\mathcal{E}'$.
- Suppose $P \equiv Q$, $P' \equiv Q'$, and $Q \xrightarrow{\mu[m]} Q'$. Then the result follows from induction and Proposition 7.4.

We then prove that if there exists a transition $\mathsf{Init} \xrightarrow{\{e\}} X$ then there exists a $P'$ and a transition $P \xrightarrow{\mu_\gamma[m]} P'$ and an isomorphism $f : \mathcal{E} \to \mathcal{E}'$ such that $\lambda(e) = \mu$, $f \circ k' = k[e \mapsto m]$, and $f(X) = \mathsf{Init}'$. We prove this by induction on $P$.

- Suppose $P = 0$. Then $E = \emptyset$, and obviously no transitions exist in $C_{br}(\mathcal{E})$.
- Suppose $P = \mathsf{roll}\ \gamma$. Then there does not exist $e \in E$ such that $\lambda(e) = \mu$.

- Suppose $P = \mathsf{rolling}\ \gamma$. Then there does not exist $e \in E$ such that $\lambda(e) = \mu$.
- Suppose $P = \alpha_\gamma.P''$. Then $\{e_\alpha\} \mapsto e'$ for all $e' \in E \setminus \{e_\alpha\}$ such that $\lambda(e) = \mu$, meaning by definition $e = e_\alpha$. In addition, by Lemma E.5, whenever $e' \in \mathsf{Init}$, $\lambda(e') \in \{\mathsf{roll}\ \gamma', \mathsf{start\ roll}\ \gamma', \mathsf{roll\ bound}\}$ meaning $\mathsf{std}(P)$. This means we get $P \xrightarrow{\alpha[m]} \alpha[m].P''$ for some fresh $m$, and the isomorphisms are similar to this case in the proof of Theorem 4.10.
- Suppose $P = \alpha[n].P''$ and $\{\!|P''|\!\} = \left\langle \mathcal{E}'', \mathsf{Init}'', k'' \right\rangle$. Then $e_\alpha \in \mathsf{Init}$, and clearly $\mathsf{Init}'' \xrightarrow{e} X''$, meaning there exists a key $m$ and a transition $P'' \xrightarrow{\lambda(e)[m]} P'''$, such that $\{\!|P'''|\!\} = \left\langle \mathcal{E}''', \mathsf{Init}''', k''' \right\rangle$ and there exists an isomorphism $f'' : \mathcal{E}'' \to \mathcal{E}'''$ such that $k'''(f''(e)) = m$ and $f''(X'') = \mathsf{Init}'''$. If $m \neq n$, then $P \xrightarrow{\lambda(e)[m]} \alpha[m].P'''$. Otherwise, we can chose a fresh $m$ and still get a transition. We define our isomorphism as $f = f''[e_\alpha \mapsto e'_\alpha]$ and the rest of the proof is straightforward.
- Suppose $P = P_0 + P_1$. Then the proof is similar to the same case in CCSK, as the choice semantics are the same.
- Suppose $P = P_0 \mid P_1$, $\{\!|P_0|\!\} = \langle \mathcal{E}_0, \mathsf{Init}_0, k_0 \rangle$, $C_{br}(\mathcal{E}_0) = (E_0, F_0, \mathsf{C}_0, \to_0)$, $\{\!|P_1|\!\} = \langle \mathcal{E}_1, \mathsf{Init}_1, k_1 \rangle$, and $C_{br}(\mathcal{E}_1) = (E_1, F_1, \mathsf{C}_1, \to_1)$. Then either $e = (Y, (e_0, *))$, $e = (Y, (*, e_1))$, or $e = (Y, (e_0, e_1))$.
  If $e = (Y, (e_0, *))$, then whenever $X'_0 \mapsto_0 e_0$, there exists $e' \in Y$ such that $\pi_0(e') \in X_0$ and $\{e'\} \mapsto e$. And whenever $\pi_0(e') \rhd_0 \pi_0(e)$, we get $e' \rhd e$. This means $\mathsf{Init}_0$ is conflict-free, $\pi_0(X)$ is conflict-free, and $\mathsf{Init}_0 \xrightarrow{e_0}_0 \pi_0(X)$. There therefore exists a key $m$ and a transition $P_0 \xrightarrow{\lambda_0(e_0)[m]} P'_0$, such that $\left\{\!\left|P'_0\right|\!\right\} = \left\langle \mathcal{E}'_0, \mathsf{Init}'_0, k'_0 \right\rangle$ and there exists an isomorphism $f_0 : \mathcal{E}_0 \to \mathcal{E}'_0$ such that $k'_0(f_0(e_0)) = m$ and $f_0(\pi_0(X)) = \mathsf{Init}'_0$.

  We chose an $m$, which is fresh for $P_1$, and we get $P \xrightarrow{\lambda_0(e_0)[m]} P'_0 \mid P_1$. We define our isomorphisms similarly to the corresponding case in the first part of the proof, and the proof of them being isomorphisms is similar.
  If $e = (Y, (*, e_1))$, the argument is similar.
  If $e = (Y, (e_0, e_1))$, then for $i \in \{0, 1\}$, whenever $X'_i \mapsto_i e_i$, there exists $e' \in Y$ such that $\pi_i(e') \in X'_i$ and $\{e'\} \mapsto e$. And whenever $\pi_i(e') \sharp_i \pi_i(e)$, we get $e' \sharp e$. This means $\mathsf{Init}_i$ is conflict-free, $\pi_i(X)$ is conflict-free, and $\mathsf{Init}_i \xrightarrow{e_0}_i \pi_i(X)$. There therefore exists a key $m_i$ and a transition $P_i \xrightarrow{\lambda_i(e_i)[m_i]} P'_i$, such that $\left\{\!\left|P'_i\right|\!\right\} = \left\langle \mathcal{E}'_i, \mathsf{Init}'_i, k'_i \right\rangle$ and there exists an isomorphism $f_i : \mathcal{E}_i \to \mathcal{E}'_i$ such that $k'_i(f_i(e_i)) = m_i$ and $f_i(\pi_i(X)) = \mathsf{Init}'_i$.
  We say that $m_0 = m_1$ is a fresh $m$, and then since $\lambda_0(e_0) = \overline{\lambda_1(e_1)}$ and $\lambda(e) = \tau$, we get $P \xrightarrow{\lambda(e)[m]} P'_0 \mid P'_1$. We define our isomorphism similarly to the corresponding case in the first part of the proof, and the proof of them being isomorphism is similar to that case. The rest of the case is straightforward.
- Suppose $P = P'' \setminus A$, $\{\!|P''|\!\} = \langle \mathcal{E}'', \mathsf{Init}, k \rangle$, and $C_{br}(\mathcal{E}'') = (E'', F'', \mathsf{C}'', \to'')$. Then $\lambda(e) \notin A \cup \overline{A}$ and there exists at least one $Y \in \mathsf{cause}(e)$ such that if $e' \in Y$ then $\lambda(e') \notin (A \cup \overline{A})$. And since $P$ is reachable, for all $e' \in \mathsf{Init}$, $\lambda(e') \notin (A \cup \overline{A})$. We therefore know $\mathsf{Init}'' = \mathsf{Init} \xrightarrow{e}{}'' X$, meaning there exists a key $n$ and a

transition $P'' \xrightarrow{\lambda''(e)} P'''$ such that $\{\!|P'''|\!\} = \left\langle \mathcal{E}''', \mathsf{Init}''', k''' \right\rangle$, and there exists an isomorphism $f' : \mathcal{E}'' \to \mathcal{E}'''$ such that $f' \circ k''' = [e \mapsto n]$ and $f'(X) = \mathsf{Init}'''$.
This means $P \xrightarrow{\lambda''(e)} P''' \setminus A$ and the isomorphism $f \restriction E$ clearly fulfils the remaining conditions.

- Suppose $P = P''[f]$, $\{\!|P''|\!\} = \langle \mathcal{E}'', \mathsf{Init}, k \rangle$. Then the case is similar to the corresponding case of Theorem 4.10.

### E.10 Lemma E.11

**Lemma E.11.** *Let $P$ be a roll-CCSK process. If $\{\!|P|\!\} = \langle (E, F, \mapsto, \rhd, \lambda, \mathsf{Act}), \mathsf{Init}, k \rangle$ where $e, e' \in E$, $e \neq e'$, and there exist $\gamma, \gamma'$ such that $\lambda(e) \in \{\mathsf{roll}\ \gamma, \mathsf{bound\ roll}\}$ and $\lambda(e') \in \{\mathsf{roll}\ \gamma', \mathsf{bound\ roll}\}$, then $e \rhd e'$.*

*Proof.* It is obvious from the syntax that $e$ and $e'$ come from parallel subprocesses, and the result follows from the parallel composition rule.

### E.11 Lemma E.12

**Lemma E.12.** *Let $P$ be a roll-CCSK process. If $\{\!|P|\!\} = \langle (E, F, \mapsto, \rhd, \lambda, \mathsf{Act}), \mathsf{Init}, k \rangle$ and $e \in E$ where $\lambda(e) \notin \{\mathsf{roll}\ \gamma, \mathsf{bound\ roll}\}$. Then whenever $X \mapsto \underline{e}$, either $X = \{e\}$ or for all $e' \in X$, $\lambda(e') \notin \{\mathsf{roll}\ \gamma, \mathsf{bound\ roll}\}$.*

*Proof.* Obvious in most cases. In parallel composition we use the fact that we never have $e'' \in X'' \mapsto e'''$ where $\lambda(e'') \in \{\mathsf{roll}\ \gamma, \mathsf{bound\ roll}\}$.

### E.12 Lemma E.13

**Lemma E.13.** *Let $P$ be a roll-CCSK process. If $\{\!|P|\!\} = \langle (E, F, \mapsto, \rhd, \lambda, \mathsf{Act}), \mathsf{Init}, k \rangle$ and $e, e' \in \mathsf{Init}$ then $e' \rhd \underline{e}$ if and only if there exist $\gamma, \gamma'$ such that $\gamma \leq_P \gamma'$, either $\lambda(e) = \mathsf{start\ roll}\ \gamma$ or $\lambda(e)_\gamma[k(e)]$ occurs in $P$, and either $\lambda(e') = \mathsf{start\ roll}\ \gamma'$ or $\lambda(e')_{\gamma'}[k(e')]$ occurs in $P$.*

*Proof.* We prove this by induction on $P$.

- It is trivial in all cases except $P = \alpha_{\gamma''}[n].P'$ and $P = P_0 \mid P_1$.
- Suppose $P = \alpha_{\gamma''}[n].P'$. Then if $e' \rhd \underline{e}$ either $e = e_\alpha$ or the result follows from induction and the fact that $e \in \mathsf{Init}$ means $\lambda(e) \notin \{\mathsf{roll}\ \gamma'', \mathsf{roll\ bound}\}$. If $e = e_\alpha$ then $e' \rhd \underline{e}$ unless $\lambda(e') = \mathsf{start\ roll}\ \gamma'$ for some $\gamma' \not\geq_P \gamma$.
  And if there exist $\gamma, \gamma'$ such that $\gamma \leq_P \gamma'$, either $\lambda(e) = \mathsf{start\ roll}\ \gamma$ or $\lambda(e)_\gamma[k(e)]$ occurs in $P$, and either $\lambda(e') = \mathsf{start\ roll}\ \gamma'$ or $\lambda(e')_{\gamma'}[k(e')]$ occurs in $P$ then either $e = e_\alpha$ or the result follows from induction. And if $e = e_\alpha$, then $\{e \mid \lambda_P(e) \in \{\mathsf{roll}\ \gamma', \mathsf{roll\ bound}\} \cup \{\mathsf{start\ roll}\ \gamma' \mid \nexists \beta, n.\beta_{\gamma'} \text{ or } \beta_{\gamma'}[n] \text{ occurs in } \alpha_\gamma.P\}\} \times \{\underline{e_\alpha}\} \subseteq \rhd$.
- Suppose $P = P_0 \mid P_1$. Then there exists $i \in \{0, 1\}$ such that either $\pi_i(e') \overline{\rhd_i}\ \underline{\pi_i(e)}$ or $\pi_i(e') = \pi_i(e)$. If $\pi_i(e') \rhd_i \underline{\pi_i(e)}$ then the result follows from induction, and if $\pi_i(e') = \pi_i(e)$ then that contradicts $e, e' \in \mathsf{Init}$.

### E.13 Proof of Theorem 7.7

*Proof.* We first prove that if there exists a $P'$ and a transition $P \overset{\rho}{\rightsquigarrow} P'$ then there exists an isomorphism $f : \mathcal{E} \to \mathcal{E}'$ and events $e_r$ and $e_0, e_1, \dots e_n$ such that $\mathsf{Init} \xrightarrow{\{e_r\}} X_0 \xrightarrow{\{e_0\}} X_1 \dots \xrightarrow{\{e_n\}} X_{n+1} \xrightarrow{\{e_r\}} X_{done}$, $\lambda(e_r) = \rho$, $\{e_0, e_1, \dots e_n\} = \{e \mid \exists \gamma'.\gamma \leq_P \gamma'$ and either $\lambda(e)_{\gamma'}[k(e)]$ occurs in $P$ or $\lambda(e) = \mathsf{start\ roll}\ \gamma'$ and $\mathsf{rolling}\ \gamma'$ occurs in $P\}$, $f \circ k' = k \upharpoonright \{e \mid f(e) \in \mathsf{Init}'\}$, and $f(X_{done}) = \mathsf{Init}'$.

We prove this through induction on the derivation of $P \overset{\mathsf{roll}\ \gamma}{\rightsquigarrow} P'$, first if $\rho = \mathsf{roll}\ \gamma$:

**(ROLL):** Suppose $P = \mathsf{rolling}\ \gamma$ and $P' = \mathsf{roll}\ \gamma$. Then $E = \{e_r, e_t\}$ and $E' = \{e_r', e_t'\}$ with the rest of $\mathcal{E}$ and $\mathcal{E}'$ as defined in the semantics, and $\mathsf{Init} = \{e_t\}$ and $\mathsf{Init}' = \emptyset$, and obviously $\mathsf{Init} \xrightarrow{\{e_r\}} \{e_r, e_t\} \xrightarrow{\{e_t\}} \{e_r\} \xrightarrow{\{e_r\}} \emptyset$, and we define $f(e) = \begin{cases} e_r' & \text{if } e = e_r \\ e_t' & \text{if } e = e_t \end{cases}$, which fulfils the conditions

**(act ROLL):** Suppose $P = \alpha_\gamma[m].R$, $P' = \alpha_\gamma.R_{\not\downarrow\{\gamma' \mid \gamma \leq_P \gamma'\}}$, $R \overset{\mathsf{roll}\ \gamma}{\rightsquigarrow} R'$, $\{\!|R|\!\} = \langle \mathcal{E}_R, \mathsf{Init}_R, k_R \rangle$, and $\left\{\!\!\left| R_{\not\downarrow\{\gamma' \mid \gamma \leq_P \gamma'\}} \right|\!\!\right\} = \langle \mathcal{E}_{R'}, \mathsf{Init}_{R'}, k_{R'} \rangle$. Then $\{\gamma' \mid \gamma \leq_P \gamma'\}$ is $\gamma$ and all $\gamma'$s for which some $\beta_{\gamma'}$ or $\beta_{\gamma'}[n]$ occurs in $P$. It is clear from the semantic rules that this means $\mathsf{Init}' = \{e \mid \lambda(e) \in \{\mathsf{start\ roll}\ \gamma' \mid \mathsf{rolling}\ \gamma'$ occurs in $P$ and $\nexists \beta, n.\beta_{\gamma'}$ or $\beta_{\gamma'}[m]$ occurs in $P\}$ and there exists an isomorphism $f : \mathcal{E} \to \mathcal{E}'$.
In addition, by induction we have $\mathsf{Init}_R \xrightarrow{\{e_r\}} X_{0_R} \xrightarrow{\{e_0\}} X_{1_R} \dots \xrightarrow{\{e_m\}} X_{m+1_R} \xrightarrow{\{e_r\}} X_{done_R}$, $\lambda(\{e_0, e_1, \dots, e_m\}) = \{\alpha_{\gamma'}[m] \mid \gamma \leq_R \gamma'$ and $\alpha_\gamma[m]'$ occurs in $R\} \cup \{\mathsf{start\ roll}\ \gamma' \mid \gamma \leq_R \gamma'$ and $\mathsf{rolling}\ \gamma'$ occurs in $R\}$ and $f(X_{done_R}) = \mathsf{Init}_{R'}$. Since $\mathsf{Init} = \mathsf{Init}_R \cup \{e_\alpha\}$, and no new preventions are added to $e_r$, we get $\mathsf{Init} \xrightarrow{\{e_r\}}$, and for all $e \in E$ such that $\lambda(e) \notin \{\mathsf{roll}\ \gamma', \mathsf{roll\ bound}\} \cup \{\mathsf{start\ roll}\ \gamma' \mid \nexists \beta, n.\beta_{\gamma'}$ or $\beta_{\gamma'}[n]$ occurs in $\alpha_\gamma.P\}$, whenever $X \mapsto \underline{e}$, we have $e_r \in X$, meaning, $\mathsf{Init} \xrightarrow{\{e_r\}} X_0 \xrightarrow{\{e_0\}} X_1 \dots \xrightarrow{\{e_n\}} X_{n+1} \xrightarrow{\{e_r\}} X_{done}$, $\lambda(e_r) = \mathsf{roll}\ \gamma$, $\lambda(\{e_0, e_1, \dots, e_n\}) = \{\alpha_{\gamma'}[m] \mid \gamma \leq_P \gamma'$ and $\alpha_\gamma[m]'$ occurs in $P\} \cup \{\mathsf{start\ roll}\ \gamma' \mid \gamma \leq_P \gamma'$ and $\mathsf{rolling}\ \gamma'$ occurs in $P\} = \mathsf{Init} \setminus \{e_r\}$.

**(par ROLL):** Suppose $P = Q \mid R$, $P' = (Q \mid R)_{\not\downarrow\{\gamma' \mid \gamma \leq_P \gamma'\}}$, $Q \overset{\mathsf{roll}\ \gamma}{\rightsquigarrow} Q'$, $\{\!|Q|\!\} = \left\langle \mathcal{E}_Q, \mathsf{Init}_Q, k_Q \right\rangle$, $\{\!|R|\!\} = \langle \mathcal{E}_R, \mathsf{Init}_R, k_R \rangle$, $\langle \mathcal{E}, \mathsf{Init}, k \rangle$ is constructed from $\left\langle \mathcal{E}_Q, \mathsf{Init}_Q, k_Q \right\rangle$ and $\langle \mathcal{E}_R, \mathsf{Init}_R, k_R \rangle$ as described in the semantics, $\left\{\!\!\left| Q_{\not\downarrow\{\gamma' \mid \gamma \leq_P \gamma'\}} \right|\!\!\right\} = \left\langle \mathcal{E}_Q', \mathsf{Init}_Q', k_Q' \right\rangle$, $\left\{\!\!\left| R_{\not\downarrow\{\gamma' \mid \gamma \leq_P \gamma'\}} \right|\!\!\right\} = \left\langle \mathcal{E}_R', \mathsf{Init}_R', k_R' \right\rangle$, and $\left\langle \mathcal{E}', \mathsf{Init}', k' \right\rangle$ is constructed from $\left\langle \mathcal{E}_Q', \mathsf{Init}_Q', k_Q' \right\rangle$ and $\left\langle \mathcal{E}_R', \mathsf{Init}_R', k_R' \right\rangle$ as described in the semantics.
It is clear from the semantics that there exists an isomorphism $f : \mathcal{E} \to \mathcal{E}'$.
In addition, by induction we have $\mathsf{Init}_Q \xrightarrow{\{e_r\}} X_{0_Q} \xrightarrow{\{e_0\}} X_{1_Q} \dots \xrightarrow{\{e_m\}} X_{m+1_Q} \xrightarrow{\{e_r\}} X_{done_Q}$, $\lambda(\{e_0, e_1, \dots, e_m\}) = \{\alpha_{\gamma'}[m] \mid \gamma \leq_R \gamma'$ and $\alpha_\gamma[m]'$ occurs in $R\} \cup \{\mathsf{start\ roll}\ \gamma' \mid \gamma \leq_R \gamma'$ and $\mathsf{rolling}\ \gamma'$ occurs in $R\}$ and $f(X_{done_R}) = \mathsf{Init}_{R'}$.

From this we get that $(e_r, *) \in E$, and for each $X \mapsto (e_r, *)$, we have an $X_Q \mapsto e_r$ such that $X = \{e \in E \mid \pi_Q(e) \in X_Q\}$, and therefore $X \cap \mathsf{Init} \neq \emptyset$. Additionally, if $e \rhd (e_r, *)$, then either $\pi_Q(e) \rhd e_r$, or $\lambda(e) \in \{\mathsf{roll}\ \gamma', \mathsf{bound\ roll}\}$, meaning $e \notin \mathsf{Init}$. We therefore get $\mathsf{Init} \xrightarrow{\{(e_r,*)\}}$. Since by Lemma E.11 $\nexists e' \in I.\lambda(e') = \mathsf{roll}\ \gamma'$ or $\mathsf{bound\ roll}$, we get that by Lemma E.12 for $e_i$, $0 \leq i \leq n$, whenever $X_i \mapsto e_i$, either $X_i = \{e_i\}$, or $e_r \in X_i$, meaning for any $e \in E$ such that $\pi_Q(e) \in \{e_0, e_1, \ldots e_n\}$, whenever $X \mapsto e$, either $X = \{e\}$ or $(e_r, *) \in X$. The rest follows from Lemma E.10 and Lemma E.13.

**(prop ROLL 1):** Suppose $P = \beta'_\gamma[m'].R$, $P' = \beta'_\gamma[m'].R'$, $\gamma' \neq \gamma$, $\{\!|R|\!\} = \langle \mathcal{E}_R, \mathsf{Init}_R, k_R \rangle$, $\{\!|R'|\!\} = \langle \mathcal{E}_{R'}, \mathsf{Init}_{R'}, k_{R'} \rangle$, and $R \overset{\mathsf{roll}\ \gamma}{\rightsquigarrow} R'$. Then by induction $\mathsf{Init}_R \xrightarrow{\{e_r\}} \xrightarrow{\{e_r\}} X_0 \xrightarrow{\{e_0\}} X_1 \ldots \xrightarrow{\{e_n\}} X_{n+1} \xrightarrow{\{e_r\}} X_{done}$, and there exists an isomorphism $f_R : \mathcal{E}_R \to \mathcal{E}_{R'}$ fulfilling the conditions. Then it is clear from the semantics that the result holds using the isomorphism $f = f_R[e_\alpha \mapsto e'_\alpha]$.

**(prop ROLL 2):** Suppose $P = \beta'_\gamma.R$, $P' = \beta'_\gamma.R'$, $\{\!|R|\!\} = \langle \mathcal{E}_R, \mathsf{Init}_R, k_R \rangle$, $\{\!|R'|\!\} = \langle \mathcal{E}_{R'}, \mathsf{Init}_{R'}, k_{R'} \rangle$, and $\mathsf{Init}_R \xrightarrow{\{e_r\}} \xrightarrow{\{e_r\}} X_0 \xrightarrow{\{e_0\}} X_1 \ldots \xrightarrow{\{e_n\}} X_{n+1} \xrightarrow{\{e_r\}} X_{done}$, and there exists an isomorphism $f_R : \mathcal{E}_R \to \mathcal{E}_{R'}$ such that $f_R(X_d) = \mathsf{Init}_{R'}$. Then it is clear from the semantics that the result holds using the isomorphism $f = f_R[e_\alpha \mapsto e'_\alpha]$.

**(prop ROLL 3):** Suppose $P = P_0 + P_1$, $P' = P'_0 + P_1$, $\{\!|P_0|\!\} = \langle \mathcal{E}_0, \mathsf{Init}_0, k_0 \rangle$, $\{\!|P_1|\!\} = \langle \mathcal{E}_1, \mathsf{Init}_1, k_1 \rangle$ $\{\!|P'_0|\!\} = \langle \mathcal{E}_{R'}, \mathsf{Init}_{R'}, k_{R'} \rangle$, and $P_0 \overset{\mathsf{roll}\ \gamma}{\rightsquigarrow} P'_0$. Then by induction $\mathsf{Init}_0 \xrightarrow{\{e_r\}} \xrightarrow{\{e_{r_0}\}} X_{0_0} \xrightarrow{\{e_{0_0}\}} X_{1_0} \ldots \xrightarrow{\{e_{n_0}\}} X_{n+1_0} \xrightarrow{\{e_{r_0}\}} X_{done_0}$, and there exists an isomorphism $f_0 : \mathcal{E}_0 \to \mathcal{E}_{0'}$ fulfilling the conditions. Then, since $P$ is consistent, $\mathsf{std}(P_1)$, and therefore $\{0\} \times \mathsf{Init}_0 \xrightarrow{\{(0,e_{r_0})\}} \{0\} \times X_{0_0} \xrightarrow{\{(0,e_{0_0})\}} \{0\} \times X_{1_0} \ldots \xrightarrow{\{(0,e_{n_0})\}} \{0\} \times X_{n+1_0} \xrightarrow{\{(0,e_{r_0})\}} \{0\} \times X_{done_0}$, and the rest obviously holds.

**(prop ROLL 4):** Suppose $P = R \setminus A$, $P' = R' \setminus A$, $\{\!|R|\!\} = \langle \mathcal{E}_R, \mathsf{Init}_R, k_R \rangle$, $\{\!|R'|\!\} = \langle \mathcal{E}_{R'}, \mathsf{Init}_{R'}, k_{R'} \rangle$, and $R \overset{\mathsf{roll}\ \gamma}{\rightsquigarrow} R'$. Then by induction $\mathsf{Init}_R \xrightarrow{\{e_r\}} \xrightarrow{\{e_r\}} X_0 \xrightarrow{\{e_0\}} X_1 \ldots \xrightarrow{\{e_n\}} X_{n+1} \xrightarrow{\{e_r\}} X_{done}$, and there exists an isomorphism $f_R : \mathcal{E}_R \to \mathcal{E}_{R'}$ such that $f_R(X_d) = \mathsf{Init}_{R'}$. Then, since $P$ is consistent, if $\alpha_\gamma[m]$ occurs in $R$, $\alpha \notin A \cup \overline{A}$, and by Thoerem 7.5, whenever $e \in \mathsf{Init}_R$, there exists $X \in \mathsf{cause}(e)$ such that $X \subseteq \rho(A \cup \overline{A})$, and the result follows.

**(prop ROLL 5):** Suppose $P = R[f]$, $P' = R'[f]$, $\gamma' \neq \gamma$, $\{\!|R|\!\} = \langle \mathcal{E}_R, \mathsf{Init}_R, k_R \rangle$, $\{\!|R'|\!\} = \langle \mathcal{E}_{R'}, \mathsf{Init}_{R'}, k_{R'} \rangle$, and $R \overset{\mathsf{roll}\ \gamma}{\rightsquigarrow} R'$. Then by induction $\mathsf{Init}_R \xrightarrow{\{e_r\}} \xrightarrow{\{e_r\}} X_0 \xrightarrow{\{e_0\}} X_1 \ldots \xrightarrow{\{e_n\}} X_{n+1} \xrightarrow{\{e_r\}} X_{done}$, and there exists an isomorphism $f_R : \mathcal{E}_R \to \mathcal{E}_{R'}$ fulfilling the conditions, and the result follows.

**(prop ROLL 6):** Suppose $P = (\nu\ \gamma')R$, $P' = (\nu\ \gamma')R'$, $\gamma' \neq \gamma$, $\{\!|R|\!\} = \langle \mathcal{E}_R, \mathsf{Init}_R, k_R \rangle$, $\{\!|R'|\!\} = \langle \mathcal{E}_{R'}, \mathsf{Init}_{R'}, k_{R'} \rangle$, and $R \overset{\mathsf{roll}\ \gamma}{\rightsquigarrow} R'$. Then by induction $\mathsf{Init}_R \xrightarrow{\{e_r\}} \xrightarrow{\{e_r\}}$

$X_0 \xrightarrow{\{\underline{e_0}\}} X_1 \ldots \xrightarrow{\{\underline{e_n}\}} X_{n+1} \xrightarrow{\{\underline{e_r}\}} X_{done}$, and there exists an isomorphism $f_R : \mathcal{E}_R \to \mathcal{E}_{R'}$ fulfilling the conditions, and the result follows.

If $\rho = \mathsf{bound\ roll}$ the proof is similar.

We then prove that if there exists events $e_r$ and $e_0, e_1, \ldots e_n$ such that $\mathsf{Init} \xrightarrow{\{e_r\}} X_0 \xrightarrow{\{\underline{e_0}\}} X_1 \ldots \xrightarrow{\{\underline{e_n}\}} X_{n+1} \xrightarrow{\{\underline{e_r}\}} X_{done}$, then there exists a $P'$ and a transition $P \overset{\rho}{\rightsquigarrow} P'$ and an isomorphism $f : \mathcal{E} \to \mathcal{E}'$ such that $\lambda(e_r) = \rho$, $\{e_0, e_1, \ldots e_n\} = \{e \mid \exists \gamma'. \gamma \leq_P \gamma'$ and either $\lambda(e)_{\gamma'}[k(e)]$ occurs in $P$ or $\lambda(e) = \mathsf{start\ roll}\ \gamma'$ and $\mathsf{rolling}\ \gamma'$ occurs in $P\}$, $f \circ k' = k \upharpoonright \{e \mid f(e) \in \mathsf{Init}'\}$, and $f(X_{done}) = \mathsf{Init}'$.

By Lemma E.12, whenever $\lambda(e) \notin \{\mathsf{roll}\ \gamma, \mathsf{roll\ bound}\}$, there exists $X$ such that $X \mapsto \underline{e}$, and for all $e' \in X$, $\lambda(e') \notin \{\mathsf{roll}\ \gamma, \mathsf{roll\ bound}\}$. Additionally, whenever $\lambda(e) \in \{\mathsf{roll}\ \gamma, \mathsf{roll\ bound}\}$, $e \notin \mathsf{Init}$. This means if $\mathsf{Init} \xrightarrow{\{e_r\}}$, then $\lambda(e) \in \{\mathsf{roll}\ \gamma, \mathsf{roll\ bound}\}$. In addition, since whenever $X \mapsto e_r$ we have $X \cap \mathsf{Init} \neq \emptyset$, it must be that a $\mathsf{rolling}\ \gamma$ occurs in $P$, such that if $\lambda(e) = \mathsf{bound\ roll}$ then $\mathsf{rolling}\ \gamma$ is bound by some $(\nu\gamma)$ and if $\lambda(e) = \mathsf{roll}\ \gamma$ then $\mathsf{rolling}\ \gamma$ is free, meaning we get $P \overset{\lambda(e_r)}{\rightsquigarrow} P'$. The rest follows from Lemma E.13.